HOMEWORK 1, MATH246C, SPRING 2018

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Exercise 11:

(1) Let $\phi_k : U_k \to \mathbb{D}$. Then, set $f := \phi_2^{-1} \circ \phi_1 : \mathbb{D} \to \mathbb{D}$. This is well defined since $U_1 \subset U_2$. Now

$$f(0) = \phi_2^{-1}(\phi_1(0)) = \phi_2^{-1}(z_0) = 0$$

and the Schwarz lemma applies to f. Hence

$$|(\phi_2)'(0)^{-1}\phi_1'(0)| = |(\phi_2^{-1})'(z_0)\phi_1'(0)| = |f'(0)| \leq 1.$$

If $|f'(0)| = 1$, then f is a rotation and hence onto. However, since $U_1 \neq U_2$, f is clearly not onto, and so the above inequality is strict. Hence $|\phi_1'(0)| \leq |\phi_2'(0)|$.

(2) Set $T_w(z) := \frac{w-z}{1-\overline{w}z}$ for $w \in \mathbb{D}$. Then T_w is a conformal self map of the disk $\mathbb{D}$. Since clearly $\phi(z) := rz + z_0$ is a conformal map from $\mathbb{D} \to D(z_0, r)$,

$$f(z) := \phi \circ T_w(z) = r\frac{w-z}{1-\overline{w}z} + z_0$$

is a conformal map $\mathbb{D} \to D(z_0, r)$. Moreover, if $w = \phi^{-1}(z_1)$, then $f(0) = z_1$. Hence, the conformal radius of $D(z_0, r)$ is

$$|f'(0)| = -r + r|w|^2 = |-r + r\frac{|z_0 - z_1|^2}{r^2}| = r - \frac{|z_0 - z_1|^2}{r}.$$

(3) By part (1), the conformal radius of U is bounded below by r. By Koebe's Quarter theorem, if $\phi : \mathbb{D} \to U$ is the conformal map which sends $0 \mapsto z_0$,

$$\phi(\mathbb{D}) \supset D(\phi(0), |\phi'(0)|/4) = D(z_0, |\phi'(0)|/4).$$

Now by the maximality of the disk of radius r centered at z_0, $D(z_0, r) \supset D(z_0, |\phi'(0)|/4)$ which implies that $|\phi'(0)| \leq 4r$.

(4) Let $\phi : \mathbb{D} \to U$ be the conformal map which sends $0 \mapsto z_0$. Without loss of generality, we may take $z_0 = 0$. Then $\phi/\phi'(0)$ is schlicht, and hence by part (iii) of Exercise 10,

$$\frac{|z|}{(1+|z|)^2} \leq \frac{|\phi(z)|}{R} \leq \frac{|z|}{(1-|z|)^2} \quad \Rightarrow \frac{R|z|}{(1+|z|)^2} \leq |\phi(z)| \leq \frac{R|z|}{(1-|z|)^2}.$$

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Or, in terms of ϕ^{-1},

$$\frac{R|\phi^{-1}(z)|}{(1+|\phi^{-1}(z)|)^2} \le |z| \le \frac{R|\phi^{-1}(z)|}{(1-|\phi^{-1}(z)|)^2}.$$

These imply that

$$\frac{|z|}{R}(1-|\phi^{-1}(z)|)^2 \le |\phi^{-1}(z)| \le \frac{|z|}{R}(1+|\phi^{-1}(z)|)^2.$$

Since ϕ^{-1} is also continuous, as $z \to 0$, $\phi^{-1}(z) \to \phi^{-1}(0) = 0$. Hence

$$\frac{|z|}{R}(1-o(1)) \le |\phi^{-1}(z)| \le \frac{|z|}{R}(1+o(1)).$$

These inequalities implies that

$$D\Big(0, \frac{\epsilon}{R}(1-o(1))\Big) \subset \phi^{-1}D(0,\epsilon) \subset D\Big(0, \frac{\epsilon}{R}(1+o(1))\Big)$$

and hence since ϕ is bijective,

$$\mathbb{D} \setminus D\Big(0, \frac{\epsilon}{R}(1-o(1))\Big) \supset \phi^{-1}\Big(U \setminus D(0,\epsilon)\Big) \supset \mathbb{D} \setminus D\Big(0, \frac{\epsilon}{R}(1+o(1))\Big)$$

Now by the monotonicity of the ring modulus, and Exercise 35 (i) from Notes 2, this shows that

$$\log\frac{R}{\epsilon} + \log(1-o(1)) \le \text{Mod}\phi^{-1}\Big(U \setminus D(0,\epsilon)\Big) \le \log\frac{R}{\epsilon} + \log(1+o(1))$$

Since the ring modulus is a conformal invariant and log is continuous with $\log(1) = 0$,

$$\log\frac{R}{\epsilon} - o(1) \le \text{Mod}\Big(U \setminus D(0,\epsilon)\Big) \le \log\frac{R}{\epsilon} + o(1)$$

Exercise 13:

(1) Since $z \mapsto \frac{z+z_n}{1+\overline{z_n}z}$ is a conformal self map of the disk,

$$g\Big(\frac{z+z_n}{1+\overline{z_n}z}\Big)$$

has the same image as g, namely $\mathbb{C}$ with two vertical rays from $i/2$ to $+i\infty$ and $-i/2$ to $-i\infty$ removed. By rescaling, and shifting we see that f_n has the claimed image. Clearly, $f_n(0) = 0$, and by a calculation

$$f_n'(0) = \frac{1}{w_n}g(z_n) < 0$$

(2) By an annoying calculation, if $f(z) = -2z/(1+z)$

$$f_n(z) - f(z) = \Big(\frac{z^2+1}{z_n} + 2\Big)\frac{z}{1-z^2} = \Big(\frac{z^2+1}{z_n} + 2\Big)g(z)$$

Now, on compact sets, g is bounded, and since $z_n = -1 + 1/n$,

$$\Big(\frac{z^2+1}{z_n} + 2\Big) \to 0 \quad \text{as } n \to \infty.$$

(3) Note that on $\mathbb{D}$,

$$\frac{z}{1+z} = \frac{1}{1+\frac{1}{z}} = \phi_2 \circ \phi_1(z)$$

where $\phi_2(z) := 1/z$, and $\phi_1(z) = z + 1$. Since ϕ_2 takes $\mathbb{D} + 1$ conformally to $\{x > 1/2\}$, and ϕ_1 takes $\mathbb{D}$ conformally to $\mathbb{D}+1$, it is clear by continuity that $\phi_2 \circ \phi_1$ takes $\mathbb{D}$ conformally to $\{x > 1/2\}$. By rescaling, f takes $\mathbb{D}$ conformally to $\{x < -1\}$.

(4) This agrees with Caratheodory's theorem, since as the slits near each other, there can be no open neighborhood centered around $-i/2$ which is disjoint from the slits for arbitrarily large n. Hence, U, the kernel of the the domains U_n is the component on one side of the slits which contains 0 (sign error above).

Exercise 17:

(1) Since the real part of p is harmonic, we have by Poisson's formula that if $z \in D(0, r) \subset \mathbb{D}$,

$$\Re p(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re\Big(\frac{re^{i\theta}+z}{re^{i\theta}-z}\Big)\Re(p(re^{i\theta})d\theta = \Re\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\theta}+z}{re^{i\theta}-z}\Re(p(re^{i\theta})d\theta\right)$$

Now, by the Cauchy-Riemann equations, holomorphic functions are defined up to an additive constant by their real part, and since the above integral is holomorphic in z,

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\theta}+z}{re^{i\theta}-z}\Re(p(re^{i\theta})d\theta + C$$

for some $C \in \mathbb{C}$. Since $p(0) = 1$, and

$$\int_0^{2\pi} \frac{re^{i\theta}+z}{re^{i\theta}-z}\Re(p(re^{i\theta})d\theta = \Re(p(0)) = 1$$

we get that $C = 0$.

(2) Define $\mu_n := (2\pi)^{-1}\Re p(r_n e^{i\theta})d\theta$, where $\{r_n\}$ is a positive strictly increasing sequence which tends to 1 from below. Now, since

$$\mu_n([0,2\pi)) = \int_0^{2\pi} (2\pi)^{-1}\Re p(r_n e^{i\theta})d\theta = p(0) = 1$$

we have that $|\mu_n| = 1$. Hence the sequence μ_n lies in the unit ball of Radon measures $\mathfrak{M}([0,2\pi)) \simeq \mathcal{C}([0,2\pi))^*$, and by Banach-Alaoglu there exists a subsequence μ_{n_j} and a measure $\mu \in \mathfrak{M}([0,2\pi))$ such that μ_{n_j} converges weak* to μ. In other words, for any continuous $f \in \mathcal{C}([0,2\pi))$,

$$\lim_{j\to\infty} \int f d\mu_{n_j} = \int f d\mu.$$

Now, let $z \in \mathbb{D}$. Pick n_J large enough such that $z \in D(0, r_{n_J}) \subset \mathbb{D}$. Then, since

$$\frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z}$$

is continuous,

$$\int_0^{2\pi} \frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z} d\mu = \lim_{j\to\infty} \int_0^{2\pi} \frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z} d\mu_{n_j}.$$

Since

$$\int_0^{2\pi} \frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z} d\mu_{n_j} = \int_0^{2\pi} \frac{r_{n_j}e^{i\theta} + z}{r_{n_j}e^{i\theta} - z} d\mu_{n_j}$$

for all $r_j \geq r_J$ by noting the integrals are path integrals which wind around one singularity at z exactly once, we have

$$\begin{aligned} p(z) &= \lim_{j\to\infty} \int_0^{2\pi} \frac{r_{n_j}e^{i\theta} + z}{r_{n_j}e^{i\theta} - z} d\mu_{n_j} \\ &= \lim_{j\to\infty} \int_0^{2\pi} \frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z} d\mu_{n_j} \\ &= \int_0^{2\pi} \frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z} d\mu \end{aligned}$$

By bounded convergence, we have

$$p(z) = \lim_{J\to\infty} \int_0^{2\pi} \frac{r_{n_J}e^{i\theta} + z}{r_{n_J}e^{i\theta} - z} d\mu = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu.$$

Now, if $\mu \in \mathfrak{M}([0,2\pi))$, define

$$f(re^{i\theta}) := A_r * \mu(\theta)$$

where

$$A_r(x) := \frac{1 + re^{-ix}}{1 - re^{-ix}}$$

Since $A_r(\theta)$ is holomorphic when regarded as a function of $x+iy = re^{i\theta}$, so is $A_r * \mu(\theta)$ by the Cauchy-Riemann equations. Moreover, $f(0) = A_0 * \mu(0) = |\mu| = 1$, and f is Herglotz.

(3) Note that in the above proof, μ is the weak* limit of the continuous functions $p_r(\theta) := p(re^{i\theta})$ as $r \to 1$, and is so uniquely defined since $\mathfrak{M}([0, 2\pi))$ is Hausdorff. To see this, note since μ is a real measure,

$$\Re p_r(\alpha) = \int_0^{2\pi} \Re \frac{1 + re^{-i(\theta+\alpha)}}{1 - re^{-i(\theta+\alpha)}} d\mu(\theta) = P_r * \mu(\alpha)$$

where

$$P_r(x) := \Re \frac{1 + re^{-ix}}{1 - re^{-ix}}$$

is the Poisson kernel. Now, for $f \in \mathcal{C}([0, 2\pi))$, and since P_r forms an approximate identity

$$\langle \Re p_r, f\rangle = \langle P_r * \mu, f\rangle = \langle \mu, P_r * f\rangle \to \langle \mu, f\rangle$$

as $r \to 1$, where $\langle , \rangle$ denotes both the standard $L^2([0, 2\pi))$ inner product and the action of μ as regarded as an element of $\mathcal{C}([0, 2\pi))^*$. Note in the above limit we have used that μ is a finite measure and the fact that $P_r * f \to f$ in L^∞. Hence μ is unique.

Exercise 18: Let $z \in \mathbb{D}$. Then pick $0 < r < 1$ such that $z \in D(0, r)$. Then as above

$$\Re p(z) = \int_0^{2\pi} \Re \frac{re^{i\theta} + z}{re^{i\theta} - z} \Re p(re^{i\theta}) d\theta.$$

Since for all θ,

$$\Re \frac{re^{i\theta} + z}{re^{i\theta} - z} \geq \frac{r - |z|}{r + |z|} \quad \Leftrightarrow \quad \Re(|z| + e^{-i\theta} z) \geq 0$$

we have

$$\Re p(z) \geq \int_0^{2\pi} \Re \frac{r - |z|}{r + |z|} \Re p(re^{i\theta}) d\theta = \frac{r - |z|}{r + |z|} \int_0^{2\pi} \Re p(re^{i\theta}) d\theta = \frac{r - |z|}{r + |z|} \Re p(0)$$

since $p(0) = 1$, this is the first inequality by taking $r \nearrow 1$. Note that trivially $\Re z \leq |z|$, so it suffices to show the upper inequality. Since by the previous exercise,

$$|p(z)| \leq \int_0^{2\pi} \left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| d|\mu|$$

and since

$$\left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| \leq \frac{1 + |z|}{1 - |z|} \quad \Leftrightarrow \quad 0 \leq |z| - \Re(e^{-i\theta} z)$$

we have

$$|p(z)| \leq \int_0^{2\pi} \frac{1 + |z|}{1 - |z|} d|\mu| = \frac{1 + |z|}{1 - |z|} |\mu|([0, 2\pi)) = \frac{1 + |z|}{1 - |z|}$$

Exercise 23:

(1) Note that for $w \in \mathbb{D}$, we have
$$\frac{1+w}{1-w} = 1 + 2\sum_{n\geq 1} w^n$$
which converges absolutely. Then, by the monotone convergence theorem and the above exercises,
$$p(z) = \int_0^{2\pi} 1 d\mu + \sum_{n\geq 1} 2\Big(\int_0^{2\pi} e^{-in\theta} d\mu\Big) z^n.$$
By the identity theorem for power series,
$$c_k = 2\int_0^{2\pi} e^{-in\theta} d\mu.$$
The lower bound for Harnack's inequality is then fundamentally proved the same as in the previous exercise: Note
$$\Re(p(z)) = 1 + 2\Re\Big(\int_0^{2\pi} \sum_{n\geq 1} (e^{-i\theta}z)^n d\mu\Big) = 1 + 2\Re\Big(\int_0^{2\pi} \frac{1}{1-e^{-i\theta}z} - 1 d\mu\Big)$$
Hence
$$\Re(p(z)) = 1 + 2\Re\Big(\int_0^{2\pi} \frac{-e^{-i\theta}z}{1-e^{-i\theta}z} d\mu\Big) \geq 1 + 2\frac{-|z|}{1+|z|}$$
which is the lower inequality. Now, for the upper inequality, if $z = e^{i\alpha}|z|$, then
$$|p(z)| \leq 1 + \sum_{n\geq 1} 2|z|^n \Big|\int_0^{2\pi} e^{i(\alpha-\theta)} d\mu\Big| \leq 1 + \sum_{n\geq 1} 2|z|^n$$
since $|e^{ix}| = |\mu| = 1$. Thus
$$|p(z)| \leq 1 + \sum_{n\geq 1} 2|z|^n = \frac{1+|z|}{1-|z|}$$

(2) This is a simple computation:
$$\Re(c_1)^2 = 4\Big(\int_0^{2\pi} \Re(e^{-i\theta}) d\theta)^2 \leq 4\int_0^{2\pi} \Re(e^{-i\theta})^2 d\theta.$$
Note that $z = x + iy$,
$$\Re(z)^2 = x^2, \qquad \Re(z^2) = x^2 - y^2$$
So
$$\Re(z)^2 \leq \frac{1}{2} + 2\Re(z^2) \quad \Leftrightarrow \quad y \leq 1.$$

Hence,

$$\Re(e^{-i\theta})^2 \leq \frac{1}{2} + 2\Re(e^{-2i\theta})$$

which implies

$$\Re(c_1)^2 \leq 4\int_0^{2\pi} \frac{1}{2} + 2\Re(e^{-2i\theta})d\mu = 2 + \Re(c_2)$$