HOMEWORK 1, MATH246C, SPRING 2018

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Exercise 5: Verify that the Riemann sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is isomorphic (as a Riemann surface) to the projective line $\mathbb{CP}^1$.

Solution 5: We have two coordinate charts (U_1, ϕ_1) and (U_2, ϕ_2) on $\mathbb{C}^*$:

$$U_1 := \mathbb{C}, \qquad U_2 := \mathbb{C}^* \setminus \{0\}$$

$$\phi_1(z) := z, \qquad \phi_2(z) := \begin{cases} \frac{1}{z}, & z \in \mathbb{C} \setminus \{0\} \\ 0, & z = \infty \end{cases}$$

We also have two coordinate charts (V_1, ψ_1) and (V_2, ψ_2) on $\mathbb{CP}^1$:

$$V_1 := \{[x, y] : x, y \in \mathbb{C}, y \neq 0\}, \qquad V_2 := \{[x, y] : x, y \in \mathbb{C}, x \neq 0\}$$

$$\psi_1([x, y]) := \frac{x}{y}, \qquad \psi_2([x, y]) := \frac{y}{x}$$

Define $f : \mathbb{CP}^1 \to \mathbb{C}^*$ by

$$f([x, y]) = \begin{cases} \frac{x}{y}, & y \neq 0 \\ \infty, & y = 0 \end{cases}$$

This map is clearly surjective and well defined. It is also injective, for if $f([x, y]) = f([x', y'])$, then either $y = y' = 0$ (and hence $[x, 0] = [x', 0]$) or $y, y' \neq 0$. In this case where $y, y' \neq 0$, $x/y = x'/y'$ and so

$$[x, y] = [x/y, 1] = [x'/y', 1] = [x', y'].$$

We have to check that the following maps are holomorphic:

$$\phi_1 \circ f \circ \psi_1^{-1}(z) = z, \qquad \text{on } \psi_1(V_1 \cap f^{-1}(U_1)) = \mathbb{C} \tag{1}$$

$$\phi_2 \circ f \circ \psi_1^{-1}(z) = \frac{1}{z}, \qquad \text{on } \psi_1(V_1 \cap f^{-1}(U_2)) = \mathbb{C} \setminus \{0\} \tag{2}$$

$$\phi_1 \circ f \circ \psi_2^{-1}(z) = \frac{1}{z}, \qquad \text{on } \psi_2(V_2 \cap f^{-1}(U_1)) = \mathbb{C} \setminus \{0\} \tag{3}$$

$$\phi_2 \circ f \circ \psi_2^{-1}(z) = z, \qquad \text{on } \psi_2(V_2 \cap f^{-1}(U_2)) = \mathbb{C} \tag{4}$$

Date: April 16, 2018.

Clearly, (1)-(4) are holomorphic, so f is a bijective holomorphism, and hence an isomorphism. □

Exercise 8:

(i) Show that all (irreducible plane projective) algebraic curves of degree 1 are isomorphic to the Riemann sphere.

(ii) Show that all (irreducible plane projective) algebraic curves of degree 2 are isomorphic to the Riemann sphere.

Solution 8:

(i) Let $P \in \mathbb{C}[X, Y, Z]$ be irreducible and of degree 1. Then, $P = aX + bY + cZ$ for some $a, b, c \in \mathbb{C}$, with not all $a, b, c = 0$. Without loss of generality, assume that $a \neq 0$. Then, we have

$$\begin{aligned} Z(P) &:= \{[z_1, z_2, z_3] : P(z_1, z_2, z_3) = 0\} \qquad (\subset \mathbb{CP}^2) \\ &= \{[z_1, z_2, z_3] : az_1 + bz_2 + cz_3 = 0\} \\ &= \{[z_1, z_2, z_3] : z_1 = -(b/a)z_2 - (c/a)z_3\} \end{aligned}$$

Note that we can explicitly make this set a Riemann surface. Consider the coordinate chart

$$U_1 := \{[1, z_2, z_3] : a + bz_2 + cz_3 = 0\}$$

Now we can assume either $b \neq 0$ or $c \neq 0$, for if both $b = c = 0$, we would have $a = 0$, contrary to our assumption, and so $U_1 = \emptyset$, and we would continue covering $Z(P)$ by the next set. So, without loss of generality, assume $b \neq 0$ and define $\phi_1 : U_1 \to \mathbb{C}$ by

$$\phi_1([1, z_2, z_3]) = z_3$$

It is clear that ϕ_1 is surjective, continuous, and by our choice of equivalence class ($z_1 = 1$), it is well defined. In addition, it is injective, for assume that $\phi_1([1, z_2, z_3]) = \phi_1([1, z_2', z_3'])$. Then, since $[1, z_2, z_3], [1, z_2', z_3'] \in U_1$,

$$a + bz_2 + cz_3 = 0, \qquad a + bz_2' + cz_3' = 0$$

Hence, since $b \neq 0$,

$$z_2 = -(c/b)z_3 - (a/b) = -(c/b)z_3' - (a/b) = z_2'.$$

Therefore, $[1, z_2, z_3] = [1, z_2', z_3']$, and ϕ_1 is a bijection. Moreover, ϕ_1 is an open map, for let $A \subset U_1$ be open. To show this, let $\zeta \in \phi_1(A)$. Then $[1, z_2, \zeta] \in A$, and since A is open, the set

$$\pi^{-1}(A) = \{(\lambda, \lambda z_2, \lambda z_3) : [1, z_2, z_3] \in A, \lambda \in \mathbb{C} \setminus \{0\}\} \subset \mathbb{C}^3$$

is open. Then, since $(1, z_2, \zeta) \in \pi^{-1}(A)$, there exists a polydisk around $(1, z_2, \zeta)$ which is contained in $\pi^{-1}(A)$. In other words, there exists a $\delta > 0$ such that $(1, z_2, \zeta') \subset \pi^{-1}(A)$ if $|\zeta' - \zeta| < \delta$. However, this implies that if $|\zeta' - \zeta| < \delta$, then $[1, z_2, \zeta'] \in A$ and hence $\zeta' \in \phi_1(A)$. Therefore, an open ball of radius δ around ζ is contained in $\phi_1(A)$. This shows that $\phi_1(A)$ is open, and so ϕ_1 is an open map. By this fact, we can conclude that ϕ_1 is in fact a homeomorphism, and the pair (U_1, ϕ_1) is a coordinate chart on $Z(P)$.

If we had instead that $b = 0$, and $c \neq 0$, we could have defined the map

$$\phi_1'([1, z_2, z_3]) = z_2$$

and done exactly as above. Likewise, we can do a similar construction for the two coordinate charts

$$U_2 := \{[z_1, 1, z_3] : az_1 + b + cz_3 = 0\}$$

$$U_3 = \{[z_1, z_2, 1] : az_1 + bz_2 + c = 0\}$$

with the coordinate maps

$$\phi_2([z_1, 1, z_3]) = z_3, \qquad \phi_3([z_1, z_2, 1]) = z_2$$

Define $f : Z(P) \to \mathbb{CP}^1$ by

$$f([z_1, z_2, z_3]) = [z_2, z_3].$$

Then, f is obviously well defined and surjective. It is also injective, for assume $f([z_1, z_2, z_3]) = f([z_1', z_2', z_3'])$. Then $z_2 = \lambda z_2'$, and $z_3 = \lambda z_3'$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Since $[z_1, z_2, z_3], [z_1', z_2', z_3'] \in Z(P)$, we have that

$$z_1 = -(b/a)z_2 - (c/a)z_3, \qquad z_1' = -(b/a)z_2' - (c/a)z_3'$$

Hence,

$$z_1 = -(b/a)z_2 - (c/a)z_3 = \lambda(-(b/a)z_2' - (c/a)z_3') = \lambda z_1'$$

and we have $[z_1, z_2, z_3] = [z_1', z_2', z_3']$. Therefore f is bijective. So, we only need to show that f is holomorphic. So, using the same notation for the coordinate charts in Exercise 5, we need to check that

$$\psi_k \circ f \circ \phi_j^{-1}(z)$$

is holomorphic on $\phi_j(U_j \cap f^{-1}(V_k))$. Lets go back to assuming that $b \neq 0$, for otherwise $U_1 = \emptyset$ and we wouldn't have to check that set. Then, we have that

$$\phi_1^{-1}(z) = [1, -(c/b)z - (a/b), z]$$

$$\phi_2^{-1}(z) = [-(c/a)z - (b/a), 1, z]$$

$$\phi_3^{-1}(z) = [-(b/a)z - (c/a), z, 1]$$

From here, it is trivial to check that the transition maps are indeed holomorphic. Hence f is an isomorphism, and by Exercise 5, $Z(P)$ is isomorphic to the Riemann Sphere.

(ii) Let $P \in \mathbb{C}[X, Y, Z]$ be a homogenous polynomial of degree 2. Then, by a theorem in linear algebra, P has the form

$$P = aX^2 + bY^2 + cZ^2$$

for some $a, b, c \in \mathbb{C}$, not all zero. Again, we can assume that $a \neq 0$. Then, since

$$aX^2 + bY^2 + cZ^2 = a\left(X - i\sqrt{\frac{b}{a}}Y\right)\left(X + i\sqrt{\frac{b}{a}}Y\right) + cZ^2$$

we can assume that P is of the form

$$aX'Y' + cZ^2$$

(This is because the mapping

$$X' = X - i\sqrt{\frac{b}{a}}Y$$

$$Y' = X + i\sqrt{\frac{b}{a}}Y$$

$$Z = Z$$

is an invertible linear transformation, and so forms an isomorphism on $Z(P)$). Normalizing, we can assume without loss of generality that

$$P = XY + \alpha Z^2$$

for some $\alpha \in \mathbb{C}$. Note that if $\alpha = 0$, then $P = XY$ is reducible, so we may also assume without loss of generality that $\alpha \neq 0$. Now, consider the map $f : Z(P) \to \mathbb{CP}^1$ given by

$$f([z_1, z_2, z_3]) = \begin{cases} \left[\frac{z_3}{z_1}, 1\right], & z_1 \neq 0 \\ [z_2, 0] & z_1 = 0 \end{cases}$$

Now, f is clearly well defined. Moreover, it is surjective, for if $[z, 1] \in \mathbb{CP}^1$,

$$f([1, -\alpha z^2, z]) = [z, 1]$$

If $z \in \mathbb{C} \setminus \{0\}$, then $[1, z] = [1/z, 1] = f([1, -\alpha/z^2, 1/z])$. If $z = 0$, then $[0, 1] = f([0, 1, 0])$. In addition, f is injective, for assume

$$f([z_1, z_2, z_3]) = f([z_1', z_2', z_3'])$$

If we are in the case where $f([z_1, z_2, z_3]) = f([z_1', z_2', z_3']) = [\cdot, 0]$, or equivalently that $z_1 = z_1' = 0$, which implies that $z_3 = z_3' = 0$ (since we are working in $Z(P)$) and obviously $z_2 = z_2'$. So, assume that we are not in this case, and so assume $z_1, z_1' \neq 0$. Then, we have that

$$\left[\frac{z_3}{z_1}, 1\right] = \left[\frac{z_3'}{z_1'}, 1\right] \qquad \Longrightarrow \qquad \frac{z_3}{z_1} = \frac{z_3'}{z_1'}.$$

Hence by rearranging,

$$z_1'^2 z_3^2 = z_3'^2 z_1^2$$

and using the relation given by being elements of $Z(P)$,

$$z_1'^2\left(\frac{-z_1z_2}{\alpha}\right) = z_1^2\left(\frac{-z_1'z_2'}{\alpha}\right) \qquad \xrightarrow{z_1,z_1'\neq 0} \qquad z_2 = z_2'$$

and f is indeed injective. Now, there exist three charts which cover $Z(P)$ and make it a Riemann surface. These charts are given by

$$U_1 := \{[z_1,z_2,z_3] : z_1z_2 + \alpha z_3^2 = 0, \quad z_1 \neq 0\}, \qquad \phi_1([1,z_2,z_3]) = z_3$$
$$U_2 := \{[z_1,z_2,z_3] : z_1z_2 + \alpha z_3^2 = 0, \quad z_2 \neq 0\}, \qquad \phi_1([z_1,1,z_3]) = z_3$$
$$U_3 := \{[z_1,z_2,z_3] : z_1z_2 + \alpha z_3^2 = 0, \quad z_3 \neq 0\}, \qquad \phi_1([z_1,z_2,1]) = z_1$$

These can easily be seen to be homeomorphisms (as done in part (i) of this exercise), and $U_1 \cup U_2 \cup U_3 = Z(P)$. Through another trivial verification, it can be seen that the transition maps

$$\psi_j \circ f \circ \phi_k$$

are indeed holomorphic on $\phi_k(U_k \cap f^{-1}(V_j))$, where (ψ_j, V_j) are the same as in Exercise 5. Hence, f is a bijective holomorphic map, and is hence an isomorphism. $\square$

Exercise 9: If a, b are complex numbers, show that the projective cubic curve

$$[z_1,z_2,z_3] : z_1^2z_3 = z_2^3 + az_2z_3^2 + bz_3^3$$

is nonsingular if and only if the discriminant $-16(4a^3 + 27b^2)$ is non-zero.

Solution 9: Equivalently, we will show that if a, b are complex numbers, the projective cubic curve

$$[z_1,z_2,z_3] : z_1^2z_3 = z_2^3 + az_2z_3^2 + bz_3^3$$

is singular if and only if the discriminant $-16(4a^3 + 27b^2)$ is zero. To this end, assume that there exist $z_1, z_2, z_3 \in \mathbb{C}$ not all zero such that

$$P = z_2^3 + az_2z_3^2 + bz_3^3 - z_1^2z_3 = 0 \tag{5}$$

$$\frac{\partial P}{\partial z_1} = -2z_1z_3 = 0 \tag{6}$$

$$\frac{\partial P}{\partial z_2} = 3z_2^2 + az_3^2 = 0 \tag{7}$$

$$\frac{\partial P}{\partial z_3} = 2az_2z_3 + 3bz_3^2 - z_1^2 = 0 \tag{8}$$

Now if $z_3 = 0$, we have a contradiction, for by (5) then $z_2 = 0$. But, if $z_2 = z_3 = 0$, then by (8) we would have that $z_1 = 0$, which is impossible since we are working in $\mathbb{C}^3 \setminus \{(0,0,0\}$. Therefore, we may assume that $z_3 \neq 0$, and by equation (6), $z_1 = 0$. Note that if $z_1 = 0$ and $a = 0$, then equation (8) implies that $b = 0$, and so the discriminant is trivially zero. In the case where $a \neq 0$, by equation again (8) we have that

$$z_2 = -\frac{3b}{2a}z_3$$

Therefore, plugging this into equation (7),

$$0 = 3\Big(-\frac{3b}{2a}z_3\Big)^2 + az_3^2 = z_3^2\Big(\frac{27b^2}{4a^2} + a\Big).$$

Since $z_3 \neq 0$, we have that

$$\frac{27b^2}{4a^2} + a = 0$$

or equivalently that the discriminant is zero.

For the other direction, assume that the discriminant is zero. If $a = 0$, then $b = 0$, and it is easy to check that if $(z_1, z_2, z_3) = (0, 0, 1)$, equations (5)-(8) vanish. Likewise, if $b = 0$ then $a = 0$, so without loss of generality, assume that $a, b \neq 0$. Let $z_1 = 0$, $z_3 = 1$, and using the above computation as inspiration, set

$$z_2 = -\frac{3b}{2a} \neq 0.$$

Then, equation (5) becomes

$$P(0, -\frac{3b}{2a}, 1) = \frac{-27b^3}{8a^3} - \frac{3}{2}b + b = \frac{-b}{8a^3}\Big(27b^2 + 4a^3\Big) = 0$$

Clearly equations (6),(7), and (8) also vanish by similar computations. Therefore, P is singular. This completes the proof in both directions. □

Exercise 14: Show that all meromorphic functions on the Riemann sphere come from rational functions as in the above example. In particular, every principal divisor on the Riemann sphere has degree zero. Give an alternate proof of this latter fact using the residue theorem.

Solution 14: Let $f \in M(\mathbb{C}^*)$. Then, f has finite zeros and finite poles, for if there were an infinite number of either, there would be an accumulation point for $\mathbb{C}^*$ is compact. Using a fractional linear transformation, we can assume that this accumulation point is not ∞, and then apply the standard power series argument to the resulting function to obtain that the function is either identically ∞ or identically 0 on $\mathbb{C}^*$. Therefore, f has a finite number poles and zeros. By considering the chart $U_1 = \mathbb{C} \subset \mathbb{C}^*$, with the map $\psi_1(z) = z$, we have by elementary complex analysis on the plane that

$$\psi_1^{-1} \circ f \circ \psi_1(z) = R(z)g(z)$$

where R is a rational function and g has no zeros or poles on $\mathbb{C}$. Without loss of generality, by applying a linear fractional transformation, we could have assume that ∞ was not a pole or zero of f. Now, on $U_2 = \mathbb{C}^* \setminus \{0\}$, with the standard map $\psi_2(z) = 1/z$, we have

$$\psi_1^{-1} \circ f \circ \psi_2(z) = f(1/z) = R(1/z)g(1/z)$$

where we define evaluation of the right hand side at ∞ in terms of a limit. At ∞, f is some finite, non-zero number. Since a pole or zero at ∞ of R corresponds to a zero or pole respectively at 0, we obtain that R has no pole at ∞, and is hence bounded and nonzero in a neighborhood of ∞. Therefore, $g(1/z)$ is bounded in a neighborhood of infinity. Using Louiville's theorem, g is seen to be constant, and hence f has the representation

$$f(z) = \alpha R(z)$$

where we evaluate $f(\infty) = \lim_{z\to 0} \alpha R(1/z)$.

Now, we shall prove that the degree of any principle divisor on the Riemann sphere is zero. Let $f \in M(\mathbb{C}^*)$. Again, since the zeros and poles of f are a finite set, there exists an N large enough such that all the zeros and poles of f which are not located at ∞ are contained in $B_N(0)$. Then, by the residue theorem for all N sufficiently large

$$\deg(f) = \sum_{P\neq\infty} \mathrm{ord}_P(f) + \mathrm{ord}_\infty(f) = \frac{1}{2\pi i}\int_{\partial B_N(0)} \frac{f'(z)}{f(z)} + \mathrm{ord}_\infty(f).$$

Since $\mathrm{ord}_\infty(f) = \mathrm{ord}_0(\tilde{f})$, where we define $\tilde{f}(z) = f(1/z)$, we have for large enough N that

$$\mathrm{ord}_0(\tilde{f}) = \frac{1}{2\pi i}\int_{\partial B_{1/N}(0)} \frac{\tilde{f}'(z)}{\tilde{f}(z)}dz = -\frac{1}{2\pi i}\int_{\partial B_{1/N}(0)} \frac{f'(1/z)}{f(1/z)}\frac{dz}{z^2}$$

Now, we can just compute that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial B_{1/N}(0)} \frac{f'(1/z)}{f(1/z)}\frac{dz}{z^2} &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(e^{-it/N}))}{f(e^{-it/N})}\frac{-ie^{-it/N}}{N}dt \\ &= \frac{-1}{2\pi i}\int_{\partial B_N(0)} \frac{f'(z)}{f(z)}dz.\end{aligned}$$

Hence, we have that

$$\deg(f) = \frac{1}{2\pi i}\int_{\partial B_N(0)} \frac{f'(z)}{f(z)}dz + \frac{-1}{2\pi i}\int_{\partial B_N(0)} \frac{f'(z)}{f(z)}dz = 0.$$

This completes the proof. $\square$

Exercise 21: Show that two divisors on the Riemann sphere are equivalent if and only if they have the same degree, so that the degree map gives an isomorphism between the divisor class group of the Riemann sphere and the integers. If D is a divisor on the Riemann sphere, show that $\dim(L(D))$ is equal to $\max(0, \deg(D)+1)$.

Solution 21: Assume that $D_1 = \sum_P c_P(P), D_2 = \sum_Q d_Q(Q)$ have the same degree. Then, define

$$f(z) := \frac{z^{-c_\infty}\prod_{P\neq\infty}(z-P)^{c_P}}{z^{-d_\infty}\prod_{Q\neq\infty}(z-Q)^{c_Q}}.$$

Now f is meromorphic on $\mathbb{C}^*$ by Exercise 14 since $\sum_P c_P - \sum_Q d_Q = 0$. Moreover,

$$(f) = (z^{-c_\infty}\prod_{P\neq\infty}(z-P)^{c_P}) - (z^{-d_\infty}\prod_{Q\neq\infty}(z-Q)^{c_Q}) = \sum c_P(P) - \sum d_Q(Q)$$

Hence, $D_1 = D_2 + (f)$, and D_1 is linearly equivalent to D_2.

Now assume that there exists an $f \in M(X)$ such that $D_1 = D_2 + (f)$. Then since f is a meromorphic function on $\mathbb{C}^*$, it has degree zero by Exercise 14. Hence

$$\deg(D_1) = \deg(D_2 + (f)) = \deg(D_2).$$

From these observations, it is clear that the degree map gives an isomorphism of $\mathrm{Div}(\mathbb{C}^*)$ and $\mathbb{Z}$.

For the final part of the exercise, assume that $D \in \mathrm{Div}(\mathbb{C}^*)$. Then by the previous arguments, for some $m \in \mathbb{Z}$ and $f \in M(\mathbb{C}^*)$,

$$D = m(\infty) + (f)$$

Hence,

$$\dim L(D) = \dim(L(m(\infty) + (f))) = \dim(L(m(\infty)))$$

So, it remains to find the dimension of $L(m(\infty))$. Let $(g) \in L(m(\infty))$. Then by definition, $\mathrm{ord}_\infty(g) \geq -m$ and $\mathrm{ord}_p(g) \geq 0$, so in particular there are no poles at points in $\mathbb{C}$. Then by Exercise 14 we have that

$$g(z) = \alpha(z-z_1)\cdots(z-z_n)$$

for some $n \in \mathbb{N}$. Now note that

$$\alpha(z-z_1)\cdots(z-z_n) = \alpha z^n + [z^{n-1}]$$

where the brackets indicate lower order terms. Thus, g has a pole of order n at ∞, so $n \leq m$. Hence

$$g \in V := \{a_0 + a_1 z + \cdots + a_m z^m : a_k \in \mathbb{C}\}$$

Since obviously $V \in L(m(\infty))$, we have that $V = L(m(\infty))$. Now V has a basis $\{1, z, \cdots, z^m\}$, so

$$\dim(L(m(\infty))) = \dim(V) = m + 1.$$

From this follows our conclusion.

Exercise 26: Let X be a compact Riemann surface, and let f, g be two non-constant meromorphic functions on X. Show that there exists a non-zero polynomial $P(z_1, z_2)$ of two variables with complex coefficients such that $P(f, g) = 0$. Show furthermore that one can take P to be irreducible.

Solution 26: Fix $f, g \in M(X)$ and $N \in \mathbb{N}$. Then define $D_N \in \mathrm{Div}(X)$ by

$$D_N = \sum_P c_P \cdot (P)$$

where the sum is over all the poles and zeros of f and g, and $c_P = c_P(N)$ is given by

$$c_P = \max_{0 \le i,j \le N} (-i\mathrm{ord}_P(f) - j\mathrm{ord}_P(g)).$$

Then, for $0 \le i, j \le N$, $f^i g^j \in M(X)$, and moreover

$$-c_P \le i\mathrm{ord}_P(f) + j\mathrm{ord}_P(g) = \mathrm{ord}_P(f^i g^j).$$

This implies $(f^i g^j) + D_N \ge 0$, and so $f^i g^j \in L(D_N)$. Now, note that

$$\deg(D_N) = \sum_P c_P \ge 0$$

Therefore by Exercise 25, part (iii),

$$\dim(L(D_N)) \le \deg(D_N) + 1$$

Looking more carefully at c_P, we see that

$$c_P = \epsilon_1 N\mathrm{ord}_P(f) + N\epsilon_2\mathrm{ord}_P(g) = N(\epsilon_1\mathrm{ord}_P(f) + \epsilon_2\mathrm{ord}_P(g))$$

for $\epsilon_k \in \{-1, 1\}$. Hence

$$\deg(D_N) = \sum_P c_P = N \sum_P \epsilon_1\mathrm{ord}_P(f) + \epsilon_2\mathrm{ord}_P(g) \in O(N)$$

so $\deg(D_N)$ grows linearly with N. Now, set V_n to be the set of all polynomials in two variables of degree less than or equal than n in each variable. The space V_n is a vector space (over $\mathbb{C}$) of dimension $(n+1)^2$. Hence, by the above argument, the set

$$\{P(f, g) : P \in V_N\} \subset L(D_N)$$

is a linear subspace. Assume that the $\{f^i g^j\}$ are linearly independent in $\{P(f, g) : P \in V_N\}$, and hence a basis for the space. Then,

$$\dim(\{P(f, g) : P \in V_N\}) = (N+1)^2.$$

However, for N large, $(N+1)^2 > \deg(D_N) + 1 \in O(N)$, which is greater than the dimension of the ambient space $L(D_N)$. So we conclude that $\{f^i g^j\}$ are not linearly independent, and thus there exists coefficients $a_{i,j}$ such that

$$P(f,g) := \sum_{0 \le i,j \le N} a_{i,j} f^i g^j = 0$$

This shows the existence of such a polynomial P.

Now, since $\mathbb{C}[X,Y]$ is a UFD, there exists a factorization

$$P(X,Y) = Q_1(X,Y) \cdots Q_m(X,Y)$$

where each Q_k is irreducible. Note that for each k, $Q_k(f,g) \in M(X) \setminus \{0\}$ and so has a finite set of zeros, for otherwise we would be done. Since

$$0 \equiv P(f,g) = Q_1(f,g) \cdots Q_m(f,g)$$

one of the Q_{k_0} must have the property that

$$Q_{k_0}(f,g) \equiv 0$$

for otherwise the union of all of the set of zero points of $Q_k(f,g)$ would still be discrete. Since we chose Q_{k_0} to be irreducible, we are done.

Exercise 28: Suppose that $(P_1) + \cdots + (P_n) - (Q_1) - \cdots - (Q_n)$ is a principal divisor on a complex torus $\mathbb{C}/\Lambda$ (we allow repetition). Show that $P_1 + \cdots + P_n - Q_1 - \cdots - Q_n = 0$ using the group law on $\mathbb{C}/\Lambda$.

Solution 28: Let f correspond to the principle divisor above. Then, we can view f as being a periodic meromorphic function on $\mathbb{C}$, with periods ω_1, ω_2, where $\langle \omega_1, \omega_2 \rangle = \Lambda$. Let P be the parallelogram in $\mathbb{C}$ which is formed by connecting $0, \omega_1, \omega_1 + \omega_2$ and ω_2. Now, by considering $P_a := a + P$ for a number $a \in \mathbb{C}$ such that P_a does not meet any poles or zeros of $\wp, \wp'$ (this is possible since the zero sets and pole sets are discrete), consider

$$\frac{1}{2\pi i} \int_{P_a} \frac{z f'(z)}{f(z)} dz.$$

Using the calculus of residues, this is equal to

$$P_1 + \cdots + P_n - Q_1 - \cdots - Q_n.$$

Moreover, using the periodicity of f around one of the boundaries of the parallelogram, we get

$$\frac{1}{2\pi i} \Big(\int_a^{a+\omega_1} - \int_{a+\omega_2}^{a+\omega_1+\omega_2} \Big) \frac{z f'(z)}{f(z)} dz = -\frac{\omega_2}{2\pi i} \int_a^{a+\omega_1} \frac{f'(z)}{f(z)} dz$$

This right hand side is $-\omega_1$ times the winding number around the origin of the closed curve $f(a \to a+\omega_1)$ (it is closed by the periodicity of f). Hence, the integral is in $\omega_1\mathbb{Z}$. Likewise, by calculating the other two opposing sides, the integral which results is in $\omega_2\mathbb{Z}$. Therefore,

$$P_1 + \cdots + P_n - Q_1 - \cdots - Q_n \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \Lambda$$

This completes the proof. $\square$.

Exercise 30: Let $\mathbb{C}/\Lambda$ be a complex torus. Show that the Weierstrass function $\wp$ obeys the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

for some complex numbers g_2, g_3 depending on Λ. Also show that the map $z \mapsto [\wp(z), \wp'(z), 1]$ for $z \in \mathbb{C}/\Lambda\backslash\{0+\Lambda\}$ (with $0+\Lambda$ mapping to $[0,1,0]$) is a holomorphic invertible map from $\mathbb{C}/\Lambda$ to the algebraic curve

$$\{[z_1, z_2, z_3] : z_2^2 z_3 = 4z_1^3 - g_2 z_1 z_3^2 - g_3 z_3^3\}$$

which is non-singular and irreducible. (Thus, every complex torus is isomorphic to an elliptic curve. The converse is also true, but will not be established here.)

Solution 30: The proof that the function $\wp$ obeys the above differential equation can be found in [Ahlfors]. Let $\Lambda = \langle \omega_1, \omega_2 \rangle$. Now, by a geometric series around the origin with $|z| < \max(|\omega_1|, |\omega_2|)$, let $\omega \in \Lambda \setminus \{0\}$

$$\frac{1}{z-\omega} = \sum_{k=0}^{\infty} \frac{z^k}{\omega^{k+1}}$$

and hence

$$\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega} = -\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots$$

for each ω and z in a neighborhood of 0. Summing over all $\omega \in \Lambda \setminus \{0\}$ and noting the convergence of the above sum is uniform,

$$\zeta(z) := \frac{1}{z} + \sum_{\omega \in \Lambda\setminus\{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega} \right) = \frac{1}{z} - \sum_{k=2}^{\infty} G_k z^{2k-1},$$

where $-\zeta'(z) = \wp(z)$ and

$$G_k = \sum_{\omega \in \Lambda\setminus\{0\}} \frac{1}{\omega^{2k}}$$

Now, differentiating the above uniformly convergent series, we obtain a Laurent expansion around 0 of $\wp$:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k-1) G_k z^{2k-2}$$

Set $g_2 := 60G_2$ and $g_3 := 140G_3$, and compute that

$$\wp'(z) - 4\wp(z)^2 + g_2\wp(z) = -g_3 + [z]$$

where $[z]$ indicates higher order terms. Since the above equation is obviously a periodic function from the left hand side, and has no poles by the right hand side, it is constant since we can view it as a holomorphic function from the torus into $\mathbb{C}$. In particular, at $z = 0$, the terms in $[z]$ are zero, so we get

$$\wp'(z) = 4\wp(z)^2 - g_2\wp(z) - g_3$$

in a neighborhood around zero. By the identity principle, the above formula hold for all z.

Define

$$P(X, Y, Z) = 4Y^3 - g_2YZ^2 - g_3Z^3 - X^2Z.$$

Then our algebraic curve above is $Z(P)$ as defined in previous exercises. Now, we show that the map $f : z \mapsto [\wp(z), \wp'(z), 1]$ for $z \in \mathbb{C}/\Lambda\setminus\{0+\Lambda\}$ and $f : z \mapsto [0, 1, 0]$ is a bijection. Let's first show its surjective, so assume that $[w_1, w_2, 1] \in Z(P)$, for the other case is obvious. Now, the differential equation above is a first-order differential equation which is induced by setting $w = \wp(z)$ has a specific solution:

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}$$

where the integration is taken over the image under $\wp$ of a path from z_0 to z which avoids the zeros and poles of $\wp'$, and where the sign of the square root is chosen such that it equals $\wp'$. This shows that $\wp$ is the inverse of an elliptic integral, and hence there is a z such that $\wp(z) = w_1$. Then, by virtue of the differential equation and the definition of the polynomial P, $\wp'(z) = w_2$. Hence, $f(z) = [w_1, w_2, 1]$, and f is surjective. To show it's injective, assume that $f(z) = f(z')$. Again, the case where $z, z' \in 0 + \Lambda$ is easy, so assume they are not in the set. Then

$$\wp(z) - \wp(z') = 0 \qquad \wp'(z) = \wp'(z')$$

Now, from Exercise 1 of [Ahlfors] on page 276, we have that for

$$\sigma(z) := z \prod \omega \in \Lambda \setminus \{0\} \Big(1 - \frac{z}{\omega}\Big) e^{(z/\omega)+(z/\omega)^2/2}$$

the following relation holds:

$$\wp(z) - \wp(z') = -\frac{\sigma(z - z')\sigma(z + z')}{\sigma^2(z)\sigma^2(z')}.$$

Hence, either $\sigma(z-z') = 0$ or $\sigma(z+z') = 0$. Since the only zero of σ is at $0 \mod \Lambda$, the case where $\sigma(z - z') = 0 \mod \Lambda$ is trivial. So assume that $\sigma(z+z') = 0$. Then $z + z' = 0 \mod \Lambda$. Because $\wp'$ is odd, we have

$$\wp'(z) = \wp'(-z') = -\wp'(z') = -\wp'(z)$$

and so z is a zero of $\wp'$. Likewise, z' is a zero of $\wp'$. Zeros of $\wp'$ have a special form due to the symmetry of $\wp$. By the periodicity and evenness of $\wp$,

$$\wp(\omega_1 - w) = \wp(w) \qquad \Longrightarrow \qquad -\wp'(\omega_1 - w) = \wp'(w).$$

Plugging in $w = \omega_1/2$, we get that $\omega_1/2$ is a zero of $\wp'$. We can likewise show that

$$\wp'(\omega_2/2) = \wp'((\omega_1 + \omega_2)/2) = 0$$

By the differential equation above, these are the distinct roots modulo the lattice of $\wp'$. Hence

$$z, z' \in \{\omega_1/2 + \Lambda, \omega_2/2 + \Lambda, (\omega_1 + \omega_2)/2 + \Lambda\} = \{e_1 + \Lambda, e_2 + \Lambda, e_3 + \Lambda\}$$

It is now clear that if z lies in one of the above cosets, then z' must lie in the same coset by the relation $z = -z' \mod \Lambda$. Hence, we have that

$$z - z' = e_k + e_k = 0 \mod \Lambda$$

Thus $z = z' \mod \Lambda$, and f is injective on $\mathbb{C}/\Lambda$.

Now, let's check that P is nonsingular. For the sake of contradiction, assume that P is singular. Now, we may assume without loss of generality that $z_1 = 0$ and $z_3 \neq 0$ by looking at $\frac{\partial P}{\partial z_2}$. Furthermore, we may assume without loss of generality that $z_3 = 1$ by homogenous coordinates. Then we have by a few calculations that

$$z_2^2 = \frac{g_2}{12}, \qquad z_2 = \frac{-3g_3}{2g_2}$$

By these relations, it is easy to compute that $\Delta = g_2^3 - 27g_3^2 = 0$. However, Δ is the discriminant of the single variable polynomial $X^3 - g_2X - g_3$. We showed above that this polynomial has roots at e_1, e_2, e_3, all distinct. Now, by some abstract algebra, a polynomial with distinct roots has a nonzero discriminant, and so we arrive at a contradiction. Hence P is nonsingular, and by Bezout, we also have that P is irreducible.

We can check that this mapping is holomorphic by using the coordinate charts $V_\alpha = \alpha + V$ on $\mathbb{C}/\Lambda$ for a sufficiently small neighborhood $V \ni 0$ and coordinate charts U_k as given in Exercise 9. Since the coordinate maps on each of these sets are just projections, it is clear we only need to check that around $f(z) = [0, 1, 0]$ we get a holomorphic function. To this end, note that for $f(z) \in Z(P) \setminus \{[0, 1, 0]\}$ we get

$$f(z) = [\wp(z), \wp'(z), 1] = [\wp(z)/\wp'(z), 1, 1/\wp'(z)]$$

Noting that the limit as $z \to z_0 \in 0 + \Lambda$, we have $\wp(z)/\wp'(z), 1/\wp'(z) \to 0$, we can obtain that the appropriate maps $V_\alpha \to \mathbb{C}/\Lambda \to Z(P) \to U_k$ is holomorphic. □

References

http://www.maths.qmul.ac.uk/~twm/MTH6140/la26.pdf

Ahlfor's book *Complex Analysis*

My classmates: John, Akash, and Ryan.