HOMEWORK 3, MATH246C, SPRING 2018

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Exercise 5: Exercise 5 Let G be a connected planar graph with $n \geq 3$ vertices. Show that the following are equivalent:

(1) G is a maximal planar graph.
(2) G has $3n - 6$ edges.
(3) Every drawing D of G divides the plane into faces that have three edges each. (This includes one unbounded face.)
(4) At least one drawing D of G divides the plane into faces that have three edges each.

Solution 5: We shall show the following implications:

$$\begin{array}{ccc} (1) & & (2) \\ \Updownarrow & \searrow \swarrow & \Uparrow \\ (3) & \Longrightarrow & (4) \end{array}$$

By inspection, this would show the statements are equivalent.

(3) $\Rightarrow$ (4) This implication is trivial.

(1) $\Rightarrow$ (4) Assume there is a face of G which has 4 edges or more. Since G is simple and planar, this face can be drawn as a polygon with 4 or more sides. Then there exist two vertexes on this polygon which can be connected without making G non-planar. Hence G is not maximal.

(3) $\Rightarrow$ (1) Assume there exists an edge e such that adding e to G results in a planar graph. This edge is then contained in a face of G. If every drawing of G in the plane results in faces which have three edges, then this added edge connects a vertex to another vertex which has already been connected to. This contradicts the simplicity of G, and so G has a drawing in which a face has 4 or more sides.

(1) $\Rightarrow$ (3) Assume there exists a face of G which has 4 or more edges. Then since G is simple and planar, this face can be drawn as a polygon with 4 or more

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edges and four or more vertices. Then for each vertex on this polygon, there exists at least one other vertex which can be connected to without destroying the simplicity of the graph. But then G is not maximal.

(4) ⇒ (2) Let D be a drawing of G in which every face is surrounded by 3 edges. Then, every face corresponds to 3 edges, and each edge corresponds to 2 faces. Hence, $2E = 3F$. Combining this with Euler's formula ($V - E + F = 2$), we get $E = 3n - 6$.

(2) ⇒ (3) Assume that $E = 3n - 6$. Then by Euler's formula, $F = 2n - 4$. Assume there exists a drawing D of G such that a face has 4 or more edges. Then since each face corresponds to 3 or more edges, and each edge corresponds to 2 faces, $2E \geq 3F$ However, since there is a face that has 4 or more edges, this inequality is strict. Hence $F < (2/3)E = 2n - 6 = F$, a contradiction. This implies that for every drawing D of G, every face has 3 edges exactly.

This completes the proof. □

Exercise 8: Deduce Theorems 3, 4 from Theorem 7.

Solution 8:

(Thm. 3) Let G be a connected planar graph. Let $\tilde{G}$ be the graph of G formed by the following process: For each face of G which has 4 or more edges, add a vertex and edges connecting this vertex to every other vertex in the face exactly once. This construction obviously terminates and results in a graph with faces which have exactly 3 edges. By Exercise 5, this implies that $\tilde{G}$ is maximal. By stereographic projection, we can consider $\tilde{G}$ as a triangulation of the Riemann sphere. By Theorem 7 there is a Riemann sphere circle packing $\mathfrak{P}$ with nerve $\mathfrak{P}^*$ which is homotopic to $\tilde{G}$.
Now, let g be a vertex in $\tilde{G}$ which is not in G. To this g corresponds a point $g^* \in \mathfrak{P}^*$ which has the property that the graph $\tilde{G} \setminus \{g\}$ is still homotopic to $\mathfrak{P}^* \setminus \{g^*\}$. Hence, the circle packing $\mathfrak{P}$ with the circle corresponding to g^* removed is a circle packing with nerve which is homotopic to $\tilde{G} \setminus \{g\}$. Continuing in this manner, we obtain a circle packing of the Riemann sphere which has nerve which is homotopic to G. By applying a linear fractional transformation, we can assume that no circle intersects the point $\infty \in \mathbb{C}^*$. Since stereographic projection sends circles on $\mathbb{C}^*$ which do not intersect ∞ to circles on $\mathbb{C}$, we can project this Riemann sphere packing to get a circle packing on the plane which is homotopic to the original circle packing on the Riemann sphere. This collection of circles on the plane shows that Theorem 3 is true.

(Thm. 4) Let G be a simple planar connected maximal graph. Then Theorem 7 implies that there exists a Riemann sphere circle packing $\mathfrak{P}$ with nerve $\mathfrak{P}^*$ which is homotopic to the graph G. Moreover, $\mathfrak{P}$ is unique up to Möbius transformations. By applying a linear fractional transformation, we can assume as in the above part that none of the circles in $\mathfrak{P}$ intersects the point $\infty \in \mathbb{C}^*$. By stereographic projection we obtain a circle packing in the plane with nerve homotopic to G. This packing is unique up to Möbius transformations, for if there was a circle packing P on $\mathbb{C}$ which has the required properties, the projection up to the Riemann sphere $\pi^{-1}(P)$ gives another circle packing. By uniqueness on $\mathbb{C}^*$, there exists a $T \in \mathcal{M}$ such that $\pi^-1(P) = T(\mathfrak{P})$. Thus $P = \pi(T(\mathfrak{P}))$, which is the circle packing we originally constructed. $\square$.

Exercise 9: Verify Theorem 7 when G has exactly four vertices.

Solution 9: Let G be a maximal planar simple graph with four vertices. Then, since the complete graph on 4 elements has $3(4) - 6 = 6$ edges, G is isomorphic (in the graph sense) to the complete graph. There clearly exists a circle packing of circles of radius $1/2$ around the third roots of unity with an additional circle lying in the center tangent to the three others. This clearly has nerve which is homotopic to G. Hence, we only need to show uniqueness.

Pick a point of tangency of any of the two circles in the above packing. By a Möbius transformation, this point can be sent to $\infty \in \mathbb{C}^*$, and the circles will be sent to two lines in $\mathbb{C}$. These two lines are parallel, for they only intersect at ∞. Moreover, these two lines are tangent to each of the two circles still left (these circles under the image of the Möbius transformation are indeed circles, for by the injectivity of the transformations, only the point of tangency is sent to infinity). Hence both circles must lie on the same side of either line and are tangent to each line. Therefore the diameter of each circle is fixed to be the distance d between the two parallel lines. Since the two circles are tangent, by a translation and rotation (Möbius transformations) we can assume the center of one of the circles lies at 0 and the other center lies on the positive real axis at 1, and $d = 1$. In this case, the parallel lines are exactly the lines $\Im z = \pm 1/2$. Since any circle packing which has nerve G can be transformed via Möbius transformations into this "canonical form", the circle packing is unique up to Möbius transformations. $\square$

Exercise 10: Let G be a maximal planar graph with at least four vertices, let v be a vertex in G, and let $v_1, \ldots, v_d$ be the neighbours of v. Show that the following are equivalent:

(i) v is non-degenerate.

(ii) The graph $G \backslash \{v, v_1, \ldots, v_d\}$is connected and non-empty, and every vertex in $v_1, \ldots, v_d$ is adjacent to at least one vertex in $G \backslash \{v, v_1, \ldots, v_d\}$.

Solution 10:

(i) $\Rightarrow$ (ii) Assume that v is a non-degenerate vertex. Then there is one other vertex other than $v, v_1, \ldots, v_d$ in G, so $G \setminus \{v, v_1, \ldots, v_d\}$ is non-empty. By maximality, for each neighbor v_k, there is a connected graph K_k which is adjacent to v_k. If $K_k = \emptyset$, then my maximality v_k is adjacent to v_{k-1}, v_{k+1}. Likewise by maximality again, it is clear that each pair of K_k, K_{k+1} are adjacent. Hence the graph $G \setminus \{v, v_1, \ldots, v_d\}$ is connected, and all the conditions of (ii) are satisfied.

(ii) $\Rightarrow$ (i) First, note that there exists one vertex outside of $G \setminus \{v, v_1, \ldots, v_d\}$ by the non-empty condition. Assume there did exist one extra adjacency e between v_j and v_k for $k \neq j-1, j, j+1$ (note that this assumption implies that $d > 3$). Without loss of generality, assume $j < k$. Then, K_k lies in one of the connected regions of the complement of the lines which connect $\{v_j, v_{j+1} \ldots, v_k\} \cup \{e\}$. Then since this region is connected and K_k is adjacent to v_k, K_k lies in one region and K_{j-1} lies in the other. But, all of the K_k are connected by assumption, a contradiction. □

Exercise 12:

(i) Show that the Poincar distance is invariant with respect to Mbius automorphisms of $D(0,1)$, thus $d(Tz_1, Tz_2) = d(z_1, z_2)$ whenever T is a transformation of the form (1). Similarly show that the hyperbolic area is invariant with respect to such transformations.

(ii) Show that the Poincar distance defines a metric on $D(0,1)$. Furthermore, show that any two distinct points z_1, z_2 are connected by a unique geodesic, which is a portion of either a line or a circle that meets the unit circle orthogonally at two points.

(iii) If C is a circle in the interior of $D(0,1)$, show that there exists a point z_C in $D(0,1)$ and a positive real number r_C (which we call the hyperbolic center and hyperbolic radius respectively) such that $C = \{z \in D(0,1) : d(z, z_C) = r_C\}$. (In general, the hyperbolic center and radius will not quite agree with their familiar Euclidean counterparts.) Conversely, show that for any $z_C \in D(0,1)$ and $r_C > 0$, the set $\{z \in D(0,1) : d(z, z_C) = r_C\}$ is a circle in $D(0,1)$.

(iv) If two circles C_1, C_2 in $D(0,1)$ are externally tangent, show that the geodesic connecting the hyperbolic centers z_{C_1}, z_{C_2} passes through the point of tangency, orthogonally to the two tangent circles.

Solution 12:

(i) Let T be given by

$$T(z) := \lambda \frac{z-\alpha}{1-al\bar{p}haz}$$

where $|\lambda| = 1$ and $\alpha \in D(0,1)$ Then

$$\begin{aligned} d(Tz_1, Tz_2) &= 2\text{arctanh}\Big|\frac{Tz_1 - Tz_2}{1 - Tz_1\overline{Tz_2}}\Big| \\ &= 2\text{arctanh}\Big|\frac{z_1 - z_2}{1 - z_1\overline{z_2}}\frac{1-\alpha\overline{z_2}}{1-\overline{\alpha}z_2}\Big| \end{aligned}$$

Since $\overline{1-\alpha\overline{z_2}} = 1 - \overline{\alpha}z_2$, the above expression is equal to $d(z_1, z_2)$. As for the invariance under area, let $E \subset D(0,1)$ be measurable. Then we can view $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the notation $T(x,y) = (T_1(x), T_2(y))$. Then since T is complex differentiable, it is real differentiable. By applying the Lebesgue change of variable formula,

$$\begin{aligned} \int_{T(E)} \frac{dzd\overline{z}}{(1-|z|^2)^2} &= \int_{T(E)} \frac{dxdy}{(1-(x^2+y^2))^2} \\ &= \int_E \frac{|\det J_T(x,y)|}{(1-(T_1(x,y)^2 + T_2(x,y)^2)^2} dxdy \end{aligned}$$

Since $|T'(z)|^2 = |\det J_T(x,y)|$ by, for example, the first chapter of Ahlfors's *Complex Analysis*, we obtain that the above is equal to

$$\int_E \Big(\frac{|T'(z)|}{(1-|T(z)|^2}\Big)^2 dzd\overline{z}.$$

By a quick calculation, it is easy to see that

$$\frac{|T'(z)|}{(1-|T(z)|^2} = \frac{1}{1-|z|^2}, \tag{1}$$

and this shows that the area measure is Möbius invariant.

(ii) To show d indeed defines a metric, we take the Riemannian manifold approach as taken in Garnett and Marshal's *Harmonic Measure*. Define

$$\rho(z_1, z_2) := \inf \int \frac{|dz|}{1-|z|^2}$$

where the infimum is taken over all smooth paths connecting z_1 to z_2. Then, by (1), the above integral is Möbius invariant. Set

$$T(z) := \frac{z - z_1}{1-\overline{z_1}z}.$$

It is clear that $T(z_1) = 0$, and so by the invariance properties,

$$\rho(z_1, z_2) = \rho(T(z_1), T(z_2)) = \rho(0, T(z_2)).$$

To calculate ρ, it suffices to calculate the shortest path to any point in $D(0,1)$ from the origin. Let $z_0 := T(z_2)$ and $\gamma : [0,1] \to D(0,1)$ be a smooth path connecting 0 and z_0. Then, there exist continuous functions r, θ such that

$$\gamma(t) = r(t)e^{i\theta(t)} \quad \Longrightarrow \quad \gamma'(t) = r'(t)e^{i\theta(t)} + ir(t)\theta'(t)e^{i\theta(t)}.$$

Then, the path integral along γ is given by

$$\begin{aligned}\int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt &= \int_0^1 \frac{\sqrt{(r'(t)e^{i\theta(t)})^2 + (r(t)\theta'(t)e^{i\theta(t)})^2}}{1-r(t)^2}dt \\ &\geq \int_0^1 \frac{|r'(t)|}{1-r(t)^2}dt \\ &= \int_{[0,z_0]} \frac{|dz|}{1-|z|^2}\end{aligned}$$

where $[0, z_0]$ is the straight radial path connecting 0 and z_0. Since the weighted integral along every path γ is bounded below by $[0, z_0]$, the shortest path is in fact obtained by $[0, z_0]$. Hence

$$\rho(z_1, z_2) = \int_{[0,T(z_2)]} \frac{|dz|}{1-|z|^2} = \frac{1}{2}\log\Big(\frac{1+|T(z_2)|}{1-|T(z_2|}\Big)$$

This expression on the right is in fact the same as

$$\operatorname{arctanh}|T(z_2)| = \operatorname{arctanh}\Big|\frac{z_2 - z_1}{1-\overline{z_1}z_2}.\Big| = \frac{1}{2}d(z_1, z_2).$$

Hence d in fact defines a metric, since ρ obviously does. Now, given two points z_1, z_2 on $D(0,1)$, we showed by the above that the geodesic connecting them is given by the image of the line from 0 to some other point in $D(0,1)$ under a Möbius transformation. Since Möbius transformations send lines and circles to lines and circles, the geodesic connecting z_1 to z_2 is either a circle or a line. By obvious planar geometry, any line which intersects the origin intersects the unit circle at right angles. Hence, since Möbius transforms are conformal maps which send $\partial D(0,1)$ to $\partial D(0,1)$, the image of a line through the origin under a Möbius transformation is a circle which intersects $\partial D(0,1)$ at the same angles as the line did. In other words, the intersection of the circle is a right angle. □

(iii) Let $C = \{z \in D(0,1) : |z - z_0| = r_0\}$ be a circle with respect to the usual Euclidean metric. Then, pick a Möbius transformation T which takes C to a circle centered at the origin. This map is a Möbius transformation, and so takes circles to circles. In this particular case, the center of the circle in the Euclidean metric lies at 0, and has radius α. Set $\beta = \tanh(\alpha)$. Then

$$B := \{z : |z| = \alpha\} = T(C)$$

and hence

$$\begin{aligned}\{z : d(z, T^{-1}(0)) = \alpha\} &= \{T^{-1}z : d(T^{-1}z, T^{-1}(0)) = \alpha\} \\ &= \{T^{-1}z : d(z, 0) = \alpha\} \\ &= \{T^{-1}z : \operatorname{arctanh}(|z|) = \alpha\} \\ &= T^{-1}(B) = C\end{aligned}$$

Likewise, let $C = \{z : d(z, z_C) = r_C\}$ be a circle in the hyperbolic metric. Then choose a Möbius transformation T which takes $z_C \to 0$. Then

$$\begin{aligned}T(C) &= \{T(z) : d(z, z_C) = r_C\} \\ &= \{z : d(T^{-1}(z), z_C) = r_C\} \\ &= \{z : d(z, Tz_C) = r_C\} \\ &= \{z : d(z, 0) = r_C\} \\ &= \{z : |z| = \tanh(r_C)\}\end{aligned}$$

Since Möbius transformations take circles to circles, C is a circle in the usual Euclidean sense. □

(iv) Let C_1 and C_2 be two externally tangent circles in $D(0, 1)$. Then by applying a Möbius transformation T, we can assume that $T(C_1)$ is centered (in the Euclidean and hyperbolic metrics) at 0 and the Euclidean center of $T(C_2)$ lies on the real axis. Then, the hyperbolic center of $T(C_2)$ lies on the real axis, and so does the point of intersection of $T(C_1)$ and $T(C_2)$. The geodesic connecting the origin and any other point in the hyperbolic metric is a straight line, as was shown above. Hence, the geodesic connecting the hyperbolic center of $T(C_1)$ and the hyperbolic center of $T(C_2)$ in the hyperbolic metric is a straight line, and obviously meets the circles at and angle of π. Then, since T^{-1} takes circles to circles and preserves angles, the geodesic connecting Z_{C_1} and Z_{C_2} intersects the tangency point of the two circles at right angles. □