HOMEWORK 4, MATH246C, SPRING 2018

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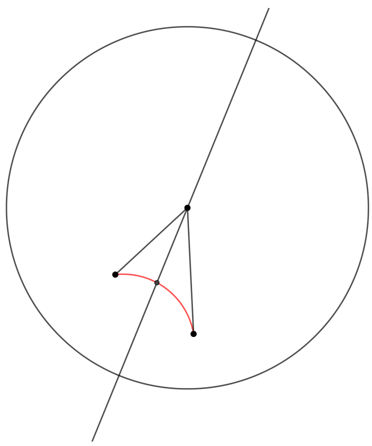
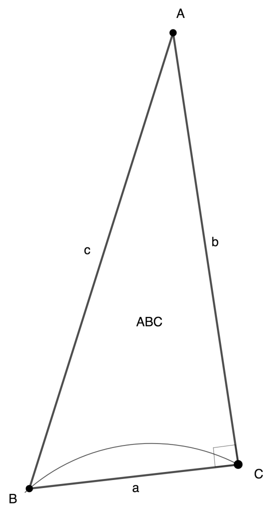
Exercise 15:

(i) Assume that $0 < r_1, r_2, r_3 < +\infty$. Then, since conformal maps preserve angles, we may assume that $p_1 = 0$ by applying a Möbius transformation. As shown in the last homework, the geodesic connecting p_1 to p_2 and p_3 is now a straight line. Therefore, we may draw a straight line whose intersection with the geodesic connecting p_2 and p_3 is at a $\pi/2$ angle (see the drawing below). Let D be the intersection of this perpendicular line and the geodesic connecting p_2 and p_3.





For the next part, we need a lemma on right hyperbolic triangles (this proof can be found at `https://www.whitman.edu/Documents/Academics/Mathematics/2014/brewert.pdf`) :

Lemma 1: Let ΔABC be a right hyperbolic triangle with the angle at C being right and none of the sides of infinite length. Denote the hyperbolic lengths of the sides of the triangles as a, b, c, where a is the side opposite

Date: May 13, 2018.

to A and so on. Then we have the following relations:

$$\sin \measuredangle A = \frac{\sinh a}{\sinh c}, \qquad \cos \measuredangle A = \frac{\tanh b}{\tanh c}$$

where $\measuredangle A$ designates the obvious angle between the two lines c and b.

Proof of Lemma: Again, without loss of generality we may assume that $A = 0$, and the lines corresponding to b and c are straight lines. Then, by Euclidean geometry, if $\overleftrightarrow{BC}$ is the straight line connecting B and C,

$$\cos \measuredangle A = \frac{|\overleftrightarrow{AC}|}{|\overleftrightarrow{AB}|} = \frac{\tanh b}{\tanh c}.$$

which is the second relation which we wanted to prove. The first is a bit more complex. Let $t := |B|$. Then B^*, the reflection of B over $\partial\mathbb{D}$ has length $1/t$, so the line segment $\overleftrightarrow{BB^*}$ has length $1/t - t = (1-t^2)/t$. Now, since

$$t = |B| = \tanh\left(\frac{d(0,B)}{2}\right) = \tanh\frac{c}{2} = \frac{e^c - 1}{e^c + 1}$$

we get that

$$e^c = \frac{1+t}{1-t}$$

and so

$$\sinh(c) = \frac{2t}{1+t^2} = |\overleftrightarrow{BB^*}|/2$$

We can likewise show that, using C and its reflection C^*,

$$\sinh(b) = |\overleftrightarrow{CC^*}|/2$$

Using the fact that the geodesic connecting B and C is an arc of a circle which intersects $\partial\mathbb{D}$ at right angles and the points C^* and B^* lie on this circle (by, say Ahlfor's Complex Analysis), we can drop midpoints on $\overleftrightarrow{BB^*}$ and $\overleftrightarrow{CC^*}$ and use some basic planar geometry to obtain that

$$\sin \measuredangle B = \frac{|\overleftrightarrow{BB^*}|}{|\overleftrightarrow{CC^*}|} = \frac{\sinh b}{\sinh c}$$

Since B was an arbitrary acute angle on the right triangle, we have showed that

$$\sin \measuredangle A = \frac{\sinh a}{\sinh c}.$$

This completes the proof of the Lemma.

These two relations actually imply, by some basic hyperbolic identities, the important relation

Corollary 1: In the same situation as in Lemma 1, we have

$$\cosh c = \cosh a \cosh b.$$

Now, recall our original triangle, with $p_1 = 0$ which has been bisected. Note that since the side of the triangle connecting p_2 and p_3 is a geodesic, $d(p_2, p_3) = d(p_2, D) + d(p_3, D)$. By Corollary 1 and a hyperbolic identity,

$$\begin{aligned}\cosh(d(0,p_2)) &= \cosh(d(p_2,D))\cosh(d(0,D)) \\ &= \cosh(d(p_2,p_3) - d(p_3,D))\cosh(d(0,D)) \\ &= \big[\cosh(d(p_2,p_3))\cosh(d(p_3,D)) - \sinh(d(p_2,p_3))\sinh(d(p_3,D))\big] \\ &\qquad \cdot \cosh(d(0,D))\end{aligned}$$

By Corollary 1 again,

$$\cosh d(0,D) = \cosh d(0,p_3)/\cosh d(p_3,D)$$

Then, by rearranging and rewriting, we have

$$\cosh d(0,p_2) = \cosh d(p_2,p_3)\cosh d(0,p_3) - \sinh d(p_2,p_3)\sinh d(0,p_3)\frac{\tanh d(p_3,D)}{\tanh d(0,p_3)}$$

Finally, by Lemma 1 we have

$$\frac{\tanh d(p_3,D)}{\tanh d(0,p_3)} = \cos\alpha_3(r_1,r_2,r_3)$$

By rearranging, we get

$$\begin{aligned}\cos\alpha_3(r_1,r_2,r_3) &= \frac{\cosh d(0,p_2) - \cosh d(p_2,p_3)\cosh d(0,p_3)}{\sinh d(p_2,p_3)\sinh d(0,p_3)} \\ &= \frac{-\cosh(r_1+r_2) + \cosh(r_2+r_3)\cosh(r_1+r_3)}{\sinh(r_2+r_3)\sinh(r_1+r_3)}.\end{aligned}$$

By permuting the sides, this implies that

$$\cos\alpha_1(r_1,r_2,r_3) = \frac{\cosh(r_1+r_2)\cosh(r_1+r_3) - \cosh(r_2+r_3)}{\sinh(r_1+r_2)\sinh(r_1+r_3)}.$$

Now, we tackle the limiting cases.

For $r_1 = \infty$, it is clear that we can assume, by a Móbius transformation that $p_2 = 0$. Then, the line connecting p_1 to 0 is a straight line which intersects the boundary of the unit disk at a right angle. Moreover, the path which connects p_1 to p_2 necessarily is an arc of a circle which meets the boundary of the unit disk at a right angle. This obviously implies $\alpha_1(\infty, r_2, r_3) = 0$.

For $r_1 = r_2 = \infty$, we can assume that $r_3 \neq \infty$. Hence moving r_3 to the origin and arguing as above, since the side of the triangle connecting p_1 to p_2 intersects the unit circle at right angles, we get $\alpha_1(\infty, \infty, r_3) = 0$.

Finally, for $r_1 = r_3 = \infty$, we can assume that $r_2 \neq \infty$. Then, the case is exactly as above, and $\alpha_1(\infty, r_2 \infty) = 0$. The limiting cases follow from taking limits of the expression we derived for $\alpha_1(r_1, r_2, r_3)$ above.

(ii) First, let's move geometries onto the upper half plane. The function $f : \mathbb{D} \to \mathbb{H}$:

$$f(z) = i\frac{1-z}{1+z}$$

is biholomorphic. We will show that

$$\text{Area}_{\mathbb{D}}(E) = \text{Area}_{\mathbb{H}}(f(E)) := \int_{f(E)} \frac{1}{y^2} dxdy.$$

By a computation, we have that

$$\Re(f(x+iy)) = \frac{2y}{(x+1)^2 + y^2}, \qquad \Im(f(x+iy)) = \frac{1 - x^2 - y^2}{(x+1)^2 + y^2}$$

and so

$$\frac{\partial \Re f}{\partial x} = \frac{-4(x+1)y}{((x+1)^2 + y^2)^2}, \qquad \frac{\partial \Re f}{\partial y} = \frac{2(x-y+1)(x+y+1)}{((x+1)^2 + y^2)^2}$$

Hence, by the Cauchy-Riemann equations the determinant of the Jacobian of f (viewed as a map $\mathbb{R}^2 \to \mathbb{R}^2$) is

$$\Big(\frac{\partial \Re f}{\partial x}\Big)^2 + \Big(\frac{\partial \Re f}{\partial y}\Big)^2 = \frac{4}{((x+1)^2 + y^2)^2}.$$

Then, by the change of variables formula,

$$\begin{aligned}
\int_{f(E)} \frac{1}{y^2} dxdy &= \int_E \frac{4}{((x+1)^2 + y^2)^2} \cdot \frac{1}{\Im(f(x+iy))^2} dxdy \\
&= \int_E \frac{4}{((x+1)^2 + y^2)^2} \cdot \frac{((x+1)^2 + y^2)^2}{(1 - x^2 - y^2)^2} dxdy \\
&= \int_E \frac{4}{(1 - x^2 - y^2)^2} dxdy \\
&= \int_E \frac{4}{(1 - |z|^2)^2} dxdy \\
&= \text{Area}_{\mathbb{D}}(E)
\end{aligned}$$

which is what we claimed. By a similar calculation, we can show that

$$d_{\mathbb{D}}(z_1, z_2) = \inf \int_\gamma \frac{1}{(1 - |z|^2)} |dz| = \int_\alpha \frac{1}{\Im(f(z))} |dz| =: d_{\mathbb{H}}(f(z_1), f(z_2))$$

and since area and length are invariant under Möbius transformations in the Poincare disk, the are also in the geometry defined above for the upper half plane. By a similar argument carried out in the previous homework on hyperbolic geodesics, we can show that, by moving points to the origin, the geodesics on $\mathbb{H}$ are given by straight vertical lines or arcs of half circles which meet the real axis.

Let T be a hyperbolic triangle (r_1, r_2, r_3) in $\mathbb{D}$ with $r_1 = \infty$. Then by a rotation, we can assume that $p_1 = -1$. Then, under f, this point gets sent to $\infty \in \mathbb{C}^*$. Now, the hyperbolics connecting p_1 to p_2 and p_1 to p_3 are sent to vertical straight lines since $p_1 \mapsto \infty$. Since $p_2, p_3 \neq -1$ they get sent to an arc of a half circle which meets the real axis at both points. We can parametrize the image of this geodesic by

$$\gamma(t) = Ce^{it}$$

where $0 \leq \theta \leq t \leq \varphi \leq \pi$. Assume that p_2 gets sent to the point $Ce^{i\theta}$. Now, this area is easy to calculate by high school calculus:

$$\begin{aligned}\text{Area}_{\mathbb{H}}(f(T)) &= \int_{C\cos(\varphi)}^{C\cos(\theta)} \int_{\sqrt{C^2-x^2}}^{\infty} \frac{1}{y^2} dy dx \\ &= \varphi - \theta\end{aligned}$$

Now, it is clear that θ is the interior angle α_2a at $f(p_2)$, and that $\pi - \varphi$ is the interior angle α_3 at $f(p_3)$. Therefore, we have that

$$\text{Area}_{\mathbb{D}}(T) = \text{Area}_{\mathbb{H}}(f(T)) = \pi - (\alpha_2 + \alpha_3)$$

Now, since f^{-1} is conformal, it preserves angles, and hence this gives the correct formula for the angles of T when viewed in $\mathbb{D}$. This completes the case when $r_1 = \infty$.

If $r_1 < \infty$, then we can extend the circular arc on which p_1 lies such that it meets $\partial\mathbb{D}$. Then, we obtain two hyperbolic triangles which meet $\partial\mathbb{D}$ whose union the previous case applies. Since the overlap of these two triangles is obviously has zero Lebesgue measure, we can add the two triangles area and the corresponding angle to get

$$\text{Area}_{\mathbb{D}}(T) = \pi - (\alpha_1 + \alpha_2 + \alpha_3)$$

(iii) This is a standard calculation. Pick a Möbius transformation T which takes $0 \mapsto p$. Then, since hyperbolic area is Möbius invariant, we only need to calculate the area of

$$T(C(p, r_0)) = \{z : |z| < \tanh(r_0/2)\}.$$

Plugging into the formula for area, we get

$$\begin{aligned}\text{Area}_{\mathbb{D}}(C(p,r_0)) &= \int_{|z|<\tanh r_0/2} \frac{4dxdy}{(1-|z|^2)^2} \\ &= \int_0^{2\pi}\int_0^{\tanh r_0/2} \frac{4rdrd\theta}{(1-r^2)^2} \\ &= 8\pi \frac{1}{2(1-u)^2}\Big|_0^{\tanh(r_0/2)} \\ &= 4\pi \sinh^2 r_0/2\end{aligned}$$

where in the last line we have used the hyperbolic identity $1-\tanh^2(x) = \sinh^2(x)$.

Exercise 18:

(i) This is a pretty awful computation. But, we can take a derivative of the expression obtained in Exercise 15 (i) to get

$$\frac{\partial\alpha_1}{\partial r_1} = \frac{-\text{csch}(r_1)\sinh(r_2+r_3+2r_1)\text{csch}(r_2+r_3+r_1)}{\sqrt{\sinh(r_2)\sinh(r_3)\sinh(r_1)\text{csch}^2(r_2+r_1)\text{csch}^2(r_3+r_1)\sinh(r_2+r_3+r_1)}}$$

Since $\text{csch}(x)$ and $\sinh(x)$ are strictly positive functions for nonzero positive values, the above expression is strictly negative, and hence α_1 is strictly decreasing in the first variable. Likewise,

$$\frac{\partial\alpha_1}{\partial r_2} = \frac{\text{csch}(r_2)\sinh(r_3+r_1)\text{csch}(r_2+r_3+r_1)}{\sqrt{\sinh(r_2)\sinh(r_3)\sinh(r_1)\text{csch}^2(r_2+r_1)\text{csch}^2(r_3+r_1)\sinh(r_2+r_3+r_1)}}$$

is strictly positive, and since α_1 is symmetric in the second and third variables, it is clear that $\partial\alpha_1/\partial r_3$ is also strictly positive.

(ii) The packing $\max(R', R'')$ is a subpacking the monotonicity proved above, since increasing the radius of any packing results in a greater central angle.

(iii) The area inequality follows from Exercise 15 (ii), and the monotonicity of increasing the first argument of α_1. Clearly, if all the radii are equal, the areas are equal, so let's assume that there exists a radius r_i which is strictly less than r_i'. Without loss of generality, $i = 1$. Now, since α_1 is strictly decreasing in the first variable and strictly increasing in the other two variables, the area of the first triangle (r_1, r_2, r_3) by Exercise 15 (ii) must be strictly less than the area of the second triangle (r_1', r_2', r_3'). The cases where some sides may be infinite follows from noting the limiting cases in Exercise 15 (i) are again strictly increasing in the first variable and strictly decreasing in the other two.

Exercise 21:

(i) By the Riemann mapping theorem, there exists a biholomorphic mapping taking the interior of the Jordan quadrilateral U to the unit disk $\mathbb{D}$. By Carathodory's theorem this extends to a mapping ϕ which is a continuous homeomorphism from $\overline{U} \to \overline{\mathbb{D}}$. Since the Jordan arc connecting p_i to p_{i+1} is connected, its image is connected in $\overline{\mathbb{D}}$. By the fact that ϕ is injective, this Jordan arc in fact is contained in $\partial\mathbb{D}$. Hence, the images of p_i lie on $\partial\mathbb{D}$ and are arranged in order (i.e. $\phi(p_1)$ is connected to $\phi(p_2)$ is connected to $\phi(p_3)$ is connected to $\phi(p_4)$ is connected to $\phi(p_1)$). By composing ϕ with a suitable rotation and then the homeomorphic map $T : \overline{\mathbb{D}} \to \overline{\mathbb{H}} \subset \mathbb{C}^*$ given by

$$z \mapsto i\frac{1-z}{1+z}$$

we get a homeomorphism $\psi : \overline{U} \to \overline{\mathbb{H}}$ which sends $p_1, \ldots, p_4$ to an increasing or decreasing sequence on $\mathbb{R}$. We can assume that this sequence is increasing by possibly multiplying ψ by -1.

Let's take a closer look at what happens to the $\phi(p_k)$ on $\partial\mathbb{D}$ when rotating $\partial\mathbb{D}$ and composing with T. It is clear that $T(e^{i\theta})$ is very large positive for $\theta < \pi$ and θ close to π. Moreover, $T(e^{i\theta})$ is very large negative for $\theta > \pi$ and θ close to π. Moreover, if $e^{i\theta}$ is outside of a neighborhood of -1, then $T(e^{i\theta})$ is bounded.

Combining these observations together, the quantity

$$\psi(p_1) + \psi(p_4) - \psi(p_2) - \psi(p_3)$$

is negative for some rotation of $\partial\mathbb{D}$ where $\phi(p_1)$ is very close to -1 but lies counterclockwise of it, and is positive for $\phi(p_4)$ very close to -1 but clockwise of it. Hence, since ψ is continuous (in $\mathbb{C}^*$) for each rotation, there exists a rotation in which

$$\psi(p_1) + \psi(p_4) - \psi(p_2) - \psi(p_3) = 0.$$

By a translation and rescaling, we can assume that $\psi(p_3) = -\psi(p_2) = 1$, and by the above equation that would imply that $\psi(p_1) = -\psi(p_4)$. This completes this part.

(ii) From the previous part, we constructed a mapping from Q to $\overline{\mathbb{H}}$ with the vertices mapping to $-r, -1, 1, r$. So it suffices to map this region to a rectangle. This can be precisely done with the Schwarz-Christoffel formula. In fact, we have an explicit map from $\mathbb{H} \to R$:

$$F(w) := \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-w^2/r^2)}}$$

This can be shown to give the correct biholomorphic map (see for example Ahlfor's Complex Analysis, page 239). In particular, using Carathodory's theorem again, we can extend this map to be a homeomorphism of $\overline{\mathbb{H}} \to \overline{R}$. Using the same connectedness argument as in the previous part of this exercise (over $\mathbb{C}^*$), the vertices p_k are connected in order. Hence, it is a vertex-preserving map.

(iii) Let R' be another rectangle which is conformally equivalent to R and Q. Then, there exist mappings which takes R, R' to $\overline{\mathbb{H}}$ by the previous exercise. Using Schwarz reflection, we can obtain a mapping from reflections over the edges of R to reflections over the edges of R'. Continuing to reflect, we obtain a map $f : \mathbb{C} \to \mathbb{C}$ which takes the tiling of $\mathbb{C}$ of R rectangles to a tiling of $\mathbb{C}$ of R' rectangles. This function is entire, and moreover grows like $O(|z|)$. To see this, let $r >> 1$, and pick a finite $n \times n$ square of R rectangles (without loss of generality centered around 0) which covers $B(0, r) \subset \mathbb{C}$. Then by construction, $f(B(0, r))$ is contained in a finite square tiling of exactly $n \times n$ R' rectangles, which we can assume to also be centered at zero. Since $R' \subset CR$ for some large C, we have that

$$f(B(0, r)) \subset CB(0, r)$$

and f grows like $O(|z|)$. Then, without loss of generality assuming that $f(0) = 0$, the function $f(z)/z$ is bounded and entire, and is hence constant. Therefore, f is affine.

Exercise 25:

(i) First, let us make some reductions. By the conformal map found in Exercise 21, we may assume without loss of generality that $Q = Q_1 \cup Q_2$ is a rectangle with vertices $0, M, i + M, i$ ordered counterclockwise. Then, it is clear that $\operatorname{mod} Q = M = |Q|$. Moreover, Q_1 and Q_2 are disjoint except at the boundary, which is a measure zero Jordan curve. Let $\alpha : [0, 1] \to Q$ be the parametrization of this curve. Note that as a result, $|Q| = |Q_1| + |Q_2|$. Let $\rho : Q_1 \to \mathbb{R}^+$ be Borel measurable. Then, for each $y \in [0, 1]$, let us define

$$f(y) := \inf\{\Re\alpha(t) : \Im\alpha(t) = y\}.$$

Now, it is clear geometrically that

$$\int_0^1 \int_0^{f(y)} \rho(x + iy)dxdy \le \int_{Q_1} \rho(x + iy)dxdy$$

and so there exists a y_0 such that

$$\int_\gamma \rho|dz| = \int_0^{f(y_0)} \rho(x + iy_0)dx \le \int_{Q_1} \rho(x + iy)dxdy$$

if γ is the parametrization of the line from $y_0 i$ to $y_0 i + f(y_0)$. Therefore, by Cauchy-Schwarz,

$$\Big(\int_\gamma \rho|dz|\Big)^2 \le \Big(\int_{Q_1} \rho dxdy\Big)^2 \le |Q_1| \int_{Q_1} \rho^2 dxdy$$

which implies that $|Q_1| \geq \mod Q_1$ by Proposition 23. Moreover, an identical argument holds for Q_2, and so we have

$$\mod Q = |Q| = |Q_1| + |Q_2| \geq \mod Q_1 + \mod Q_2$$

which is what we want.

Now, assume that equality holds in the line above. Then clearly we must have that $|Q_k| = \mod Q_k$ by the above bounds. Let $n \in \mathbb{N}$. Then, by Proposition 23, there exists a Borel measurable positive function ρ_n such that for all γ connecting the a-sides of Q_1,

$$\Big(\int_\gamma \rho_n|dz|\Big)^2 > (|Q_1| - 1/n) \int_{Q_1} \rho_n^2 dxdy.$$

We can in fact chose $\rho_n \in L^2(Q_1)$ and $\|\rho_n\|_{L^2(Q_1)} = 1$ by noting that the inequality in Proposition 23 is scale invariant, and is trivially true if $\rho_n \notin L^2(Q_1)$. Then, we have that for any γ,

$$\begin{aligned}\Big(\int_\gamma \rho_n|dz|\Big)^2 &> (|Q_1| - 1/n) \int_{Q_1} \rho_n^2 dxdy \\ &= |Q_1| \int_{Q_1} \rho_n^2 dxdy - 1/n\end{aligned}$$

Now, let $B \subset Q_1$ be the rectangle whose right side is just tangent to Q_1. Approximating B by a slightly smaller rectangle B', we can form a family of curves which are lines parallel to the real axis on B' and linear in $B \backslash B'$ as to hit the point of tangency of B and Q_1. As B' more closely approximates B, the images of the curves will have small complement with B (it will be two right triangles with one fixed side and one decreasing side). Then, by the pigeonhole principle we have that there exists a γ_n such that

$$\Big(\int_{\gamma_n} \rho_n|dz|\Big)^2 \le \Big(\int_B \rho_n dxdy\Big)^2 + 1/n$$

We have in total that

$$|Q_1| = |Q_1| \int_{Q_1} \rho_n^2 dxdy \le \Big(\int_B \rho_n dxdy\Big)^2 + 2/n$$

By Cauchy-Schwarz,

$$\Big(\int_B \rho_n dxdy\Big)^2 \le |B| \int_Q \rho_n^2 dxdy = |B|$$

Hence for all $n \in \mathbb{N}$

$$|Q_1| \leq |B| + 2/n$$

which shows that $|Q_1 \setminus B|$ has measure 0. Since the curve bounding the right side of Q_1 is continuous, its image is closed. If there was any point where this curve did not agree with the vertical line segment bounding the right side of B, the difference $Q_1 \setminus B$ would be open, and hence have positive measure. Therefore $Q_1 = B$. Note that this also implies that Q_2 is a rectangle.

(ii) This follows from cyclically permuting the indices of Q_1, Q_2 in the above, which relates to gluing $Q_1 \cup Q_2$ along the b-side rather than the a-side. Note that when cyclically permuting the indices, the modulus of Q_k is inverted. Hence we have

$$\text{mod } Q_1 \cup Q_2 \geq \frac{1}{\text{mod } Q_1} + \frac{1}{\text{mod } Q_2}.$$

References: I am indebted to John Hostle, Akash Kulkarni, and Van Latimer in particular for the solution to the last problem. I have used James W. Anderson's book on Hyperbolic Geometry, and the notes on Hyperbolic trigonometry by Tiffani Traver which can be found at: `https://www.whitman.edu/Documents/Academics/Mathematics/2014/brewert.pdf` for Exercise 15.