HOMEWORK 6, MATH246C, SPRING 2018

JOSEPH K. MILLER

Exercise 36: This proof can be found in Marshal's (to be published) book Complex Analysis. Let A be a ring domain with C_1 be the inner ring , and C_2 be the outer ring. Then, the area enclosed by C_2 is a simply connected region, and so by the Riemann mapping theorem, there exists a conformal mapping ϕ_1 which sends the area enclosed by C_2 to $\mathbb{D}$. By an appropriate inversion through a point which $\phi_1(C_1)$ winds around, we may assume without loss of generality that A consists of $C_1 = \partial\mathbb{D}$ as the inner boundary, and a Jordan curve C_2 as it's outer boundary. Applying the Riemann mapping theorem again, there exists a conformal map ϕ_2 which maps the area enclosed by the Jordan curve C_2 to $\mathbb{D}$. Then, since ϕ_2 is analytic, $\phi_2(\partial\mathbb{D})$ is an analytic curve, so we finally assume without loss of generality that A is has inner boundary C_1 an analytic curve, and $C_2 = \partial\mathbb{D}$.

Now, by Theorem 2.3 of Chapter XIII of Marshal's book Complex Analysis (or the corresponding theorem in Garnett and Marshall's book Harmonic Measure), A is a regular set, and so there exists a solution ω to the Dirichlet problem with continuous function

$$f(z) = \begin{cases} 0, & z \in C_2 = \partial\mathbb{D} \\ 1 & z \in C_1. \end{cases}$$

By the Schwarz reflection principle, we can extend ω to be harmonic and defined on an open set $U \supset \overline{A}$. Now, by the Cauchy-Riemann equation requirement for holomorphicity, and the fact that harmonic functions are analytic, the function $h := \omega_x - i\omega_y$ is a holomorphic function on U. Now, if $\mathbf{n}$ is the inward pointing normal to $\partial\mathbb{D}$, we have

$$\frac{\partial\omega}{\partial\mathbf{n}} \leq 0.$$

This is because, by the maximum principle, $0 < \omega < 1$ on A, and $\omega \equiv 1$ on $\partial\mathbb{D}$, which implies that ω is decreasing in the $\mathbf{n}$ direction. Moreover, by the maximum principle, since ω is not constant on U, it cannot attain it's maximum on $\partial\mathbb{D}$, so the above inequality is strict. Therefore, we have

$$C := i\int_{\partial\mathbb{D}} h(z)dz = -\int_{\partial\mathbb{D}} \frac{\partial\omega}{\partial\mathbf{n}} ds > 0.$$

Now, let L be an arbitrary straight line connecting C_1 to C_2 (these exist by taking intersections points of a line through the origin and C_1, C_2 which minimize the

Date: May 23, 2018.

distance). Now, by possibly shrinking U, the set $U \setminus L$ is simply connected, and for any fixed $z_0 \in A \setminus L$, the function defined by

$$H(z) := \int_{\gamma: z_0 \rightsquigarrow z} h(z)dz$$

is well defined for any $z \in U \setminus L$. Moreover, by our remark above, the function

$$g(z) := \exp(2\pi H/C)$$

can be extended to an analytic function on U. To see this, note that we only need to show that the path integral in the definition of H gives a well defined result for g over any path $\gamma \subset U$ which starts at z_0.

To this end, let $\alpha, \beta \subset U$ be arbitrary paths in U which start at z_0. Then, the path $\alpha - \beta$ is homologous to $k\partial\mathbb{D}$ for some $k \in \mathbb{Z}$. In this case, Cauchy's theorem gives that

$$\int_{\alpha-\beta} ih(z)dz = \int_{k\partial\mathbb{D}} ih(z)dz = kC \quad \Rightarrow \int_{\alpha} h(z)dz = \frac{kC}{i} + \int_{\beta} h(z)dz$$

and so we the expression

$$\exp(2\pi H/C)$$

is well defined. Note

$$|g(z)| = \exp(2\pi\omega(z)/C)) = \begin{cases} 1, & z \in C_1 \\ \exp(2\pi/C), & z \in C_2. \end{cases}$$

Since the argument of g increases by 2π winding along $\partial\mathbb{D}$ in the positive counterclockwise orientation, and decreases by 2π winding along C_1 in the negative clockwise orientation, we have by the argument principle that g attains every value in $\{1 < |z| < \exp(2\pi/C)\}$ exactly once on A. Hence, A is conformally equivalent to a circular annulus.

Exercise 39: Let $V := \bigcup V_n \subset \overline{B(0,R)}$. Now, by Arzelá-Ascoli, it suffices to show that for any compact K in U, the sequence $\{\phi_n\}$ is uniformly bounded and equicontinuous. Since $\overline{B(0,R)}$ is bounded, it suffices to show that $\{\phi_n\}$ is equicontinuous. For the sake of contradiction, assume it's not.

Then, there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exist $n = n(\delta) \in \mathbb{N}$, $z_1 = z_1(\delta) \in K$, and $z_2 = z_2(\delta) \in K$ such that $|z_1 - z_2| < \delta$ and $|\phi_n(z_1) - \phi_n(z_2)| \geq \epsilon$. Define $r := \text{dist}(K, \partial U) > 0$. Then, for each $r/2 > \delta > 0$,

$$z_1, z_2 \in B(z_1, \delta) \subset B(z_1, r/2).$$

So, for all $r/2 > \delta > 0$, the points z_1, z_2 are contained in the inner region of the annulus

$$A_\delta := \{\delta < |z - z_1| < r/2\}.$$

Define $s := \text{diam}(\phi_n(K)) \leq R$. Then, the points $\phi_n(z_1)$ and $\phi_n(z_2)$ are contained in the set

$$B(\phi_n(z_1), 2R)$$

and are hence contained in the inner region of the ring domain

$$B_\delta := B(\phi_n(z_1), 2R) \setminus [\phi_n(z_1), \phi_n(z_2)].$$

Now, since $|\phi_n(z_1), \phi_n(z_2)| \geq \epsilon$ for all $\delta > 0$,

$$B_\delta \subset \{\epsilon/2 < |\phi_n(z_1) - \phi_n(z)| < 2R\}.$$

By Exercise 35 (i) and (iii), we have

$$\text{Mod}(B_\delta) \leq \text{Mod}(\{\epsilon < |\phi_n(z_1) - \phi_n(z)| < 2R\}) = \log(4R/\epsilon) < \infty$$

for all $\delta > 0$. Now since $z_1, z_2 \notin A_\delta$, we have that $\phi_n(z_1), \phi_n(z_2) \notin \phi_n(A_\delta) \subset B_\delta$. By renormalization, we can apply Grötsch's theorem to obtain that

$$\text{Mod}(\phi_n(A_\delta)) \leq \text{Mod}(B_\delta).$$

Since ϕ_n is K-quasiconformal,

$$\frac{1}{K}\text{Mod}(A_\delta) \leq \text{Mod}(\phi_n(A_\delta))$$

and so

$$\text{Mod}(A_\delta) \leq K \log(4R/\epsilon).$$

By Exercise 35 (i) once more,

$$\text{Mod}(A_\delta) = \log(r/2\delta)$$

and therefore

$$\log(r/2\delta) \leq K \log(4R/\epsilon) < \infty$$

for all $\delta > 0$. However, since R, ϵ and r do not depend on δ, we get a contradiction by sending $\delta \searrow 0$. This shows that $\{\phi_n\}$ must be equicontinuous.

Exercise 43: Let p be an intersection of the circles C_1 and C_2. By the Möbius transformation $T : z \mapsto 1/(z - p)$, these circles get sent to parallel lines which are tangent to $T(C_3)$. By continuity, T sends the interstice of C_1, C_2, C_3 to one of the two regions bounded by $T(C_3)$ and the two parallel lines $T(C_1), T(C_2)$.

This is because the interstice has boundary points on the arcs of C_1, C_2 and C_3 and so the image of the interstice must have boundary points corresponding to the image of these arcs. Thus, the interstice must equal to the union of some of the five regions cut out by the image of the circles. By noting that $p \mapsto \infty$, we can exclude the region internal to the circle $T(C_3)$. By noting that T preserves orientations, the interstice can be mapped into two disjoint regions; the one bounded outside of the lines $T(C_1)$, $T(C_2)$ and the ones bounded within the lines. Since the region bounded outside of the lines $T(C_1)$ and $T(C_2)$ only has two boundary points on $T(C_3)$, the interstice must map to the region bounded within the lines.

Since there are points on C_3 which are not boundary points of the interstice, it must be precisely one of these regions. Since T is orientation preserving, it lies in the region which is determined by the relative ordering of the circles.

By rescaling, rotating, and translating, we can find a map S which takes the circle C_1 to the line $\Im z = -1$, the circle C_2 to $\Im z = 1$, and the circle C_3 to $\partial\mathbb{D}$. Likewise, we can find a map S' which takes C_1', C_2', and C_3' to the same lines and circles and whose interstice maps to the same region. Note that since C_1, C_2, C_3 and C_1', C_2', C_3 have the same relative orientation, the interstice of each collection of circles is mapped to the same region by S and S'. Then, the map $(S')^{-1} \circ S$ is the conformal map we are looking for.

Exercise 46: We want to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$U \subset (U_\delta)^\epsilon := \bigcup_{z \in U_\delta} \{w : |w - z| < \epsilon\}.$$

To show this, let $\epsilon > 0$, note that since U is bounded there exists a compact set $K \subset U$ and satisfies

$$U \subset K^{\epsilon/2}.$$

Pick $\delta_0 > 0$ small enough such that $U_{\delta_0} \neq \emptyset$. Then, for $z \in K$, there exists a path γ_z connecting the vertex of a triangle in U_{δ_0} to z. The image of γ_z is compact, and doesn't intersect U, so there is a η_z-fattening of the image of γ_z which is completely contained in U. Then, by taking δ_z small enough, there is clearly a series of flowers corresponding to the triangles in U_{δ_z} which covers the image of γ_z and z. Let B_δ be the collection of flowers corresponding to U_δ By the compactness of K, there is a finite number n such that

$$\bigcup_{k=1}^{n} B_{\delta_{z_n}} \supset K.$$

Since the sets B_δ are clearly decreasing in δ, this implies that if $\delta := \min_{k=1,\ldots,n}(\delta_{z_k})$,

$$B_\delta \supset K.$$

Moreover, if we take $\delta' < \min(\delta', \epsilon/2)$, we have

$$(U_{\delta'})^{\epsilon/2} \supset K.$$

Hence, we have found a $\delta' > 0$ such that

$$K \subset (U_{\delta'})^{\epsilon/2}, \quad \text{and} \quad U \subset K^{\epsilon/2}.$$

This implies, by the triangle inequality, that

$$U \subset (U_{\delta'})^\epsilon$$

which is what we wanted to show.

Acknowledgments: Thanks to Akash Kulkarni, John Hosle, and Van Latimer for some ideas and hints.