HOMEWORK 7, MATH246C, SPRING 2018

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Notes 2, Exercise 47: The first part of this exercise is the Length-Area Lemma as found in Rodin and Sullivan's paper on the convergence of circle packings to the Riemann mapping. Let C be an arbitrary circle in $\mathcal{C}'_\epsilon$ with radius R. Then, let $S_1, S_2, \ldots, S_k$ be k disjoint chains which separate C from the origin and a point on $\partial\mathbb{D}$. Assume that the chain S_i is comprised of circles $c_{i,j}$ with radii $r_{i,j}$. Also, let n_i be the number of circles in the ith chain S_i. Then, by the Schwarz inequality,

$$\Big(\sum_j r_{i,j}\Big)^2 \leq n_i \sum_j r_{i,j}^2$$

Then, if $s_i = 2\sum_j r_{i,j}$ is the geometric length of S_i,

$$n_i^{-1} s_i^2 \leq 4 \sum_j r_{i,j}^2 \quad \Rightarrow \quad \sum_i n_i^{-1} s_i^2 \leq 4 \sum_{i,j} r_{i,j}^2$$

Since $\pi r_{i,j}^2$ is the area of $c_{i,j}$, we have that

$$4 \sum_{i,j} r_{i,j}^2 \leq 4\pi$$

and so

$$\sum_i n_i^{-1} s_i^2 \leq 4\pi$$

By taking $s := \min_i s_i$, we get

$$s^2 \sum_i n_i^{-1} \leq 4\pi.$$

Now since s is greater than or equal to the diameter of C, which is in turn equal to $2R$,

$$(2R)^2 \sum_i n_i^{-1} \leq 4\pi$$

or equivalently

$$R \leq \pi^{1/2} \Big(\sum_i n_i^{-1}\Big)^{-1/2}.$$

Now, let $\delta > 0$. Pick $\epsilon > 0$ so small that there are N chains in $\mathcal{C}'_\epsilon$ which separate C from 0 and $\partial\mathbb{D}$, for N to be determined momentarily. This is possible, since

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the number of circles contained in $\mathcal{C}'_\epsilon$ is unbounded as $\epsilon \to 0$. Then, since it is a hexagonal circle packing, each chain S_i has length $\leq 6i$. Then,

$$R \leq \pi^{1/2}\Big(\sum_{i=1}^{N} n_i^{-1}\Big)^{-1/2} \leq \Big(\frac{1}{6\pi}\Big)^{-1/2}\Big(\sum_{i=1}^{N}\frac{1}{i}\Big)^{-1/2}.$$

Since $\sum_{i=1}^{N} \leq C\log(N)$ for large N and some fixed C which depends only on the behavior of log,

$$R \leq \Big(\frac{1}{6\pi}\Big)^{-1/2} C\log^{-1/2}(N).$$

Since $\log^{-1/2}(N) \to 0$ as $N \to \infty$, we could have picked N so large that

$$R < \delta.$$

Since C was an arbitrary circle in $\mathcal{C}'_\epsilon$, we have that the circles in $\mathcal{C}'_\epsilon$ uniformly decrease in radii.

This show that D_ϵ converges to $\mathbb{D}$ in the Hausdorff sense. This follows from the observation that, given $\delta > 0$, the border circles of D_ϵ which are tangent to $\mathbb{D}$ have centers which lie within ϵ of $\partial\mathbb{D}$. Since each consecutive boundary circles are tangent to each other, the maximum distance a triangle in D_ϵ can be from $\partial\mathbb{D}$ is $O(\epsilon)$.

Notes 3, Exercise 3: This is an application of Fourier series. By the change of variables formula,

$$|f(\mathbb{D})| = \int_{f(\mathbb{D})} 1dxdy = \int_{\mathbb{D}} |f'(z)|^2 dxdy = \int_{\mathbb{D}} f'(z)\overline{f'(z)}dxdy$$

For $0 < R < 1$, consider

$$A(R) := \int_0^{2\pi}\int_0^R f'(z)\overline{f'(z)}rdrd\theta \leq |f(\mathbb{D})|$$

Now, since f is holomorphic, the power series $\sum_n na_nz^{n-1}$ of f' converges absolutely, and hence the following manipulation is valid

$$\int_0^{2\pi}\int_0^R f'(z)\overline{f'(z)}rdrd\theta = \int_0^R\int_0^{2\pi}\sum_j\sum_k jka_j\overline{a_k}r^{j-1}r^{k-1}re^{i\theta(j-k)}d\theta dr$$

Then, by a computation,

$$\begin{aligned}\sum_j\sum_k\int_0^R\int_0^{2\pi} jka_j\overline{a_k}r^{j-1}r^{k-1}re^{i\theta(j-k)}d\theta dr &= \sum_j\sum_k\Big(\int_0^{2\pi} e^{i\theta(j-k)}d\theta\Big)\int_0^R jka_j\overline{a_k}r^{j-1}r^{k-1}rdr\\ &= \sum_j\sum_k(\delta_{j-k})\int_0^R jka_j\overline{a_k}r^{j-1}r^{k-1}rdr\\ &= \sum_n\int_0^R n^2a_n\overline{a_n}r^{n-1}r^{n-1}rdr\\ &= \pi\sum_n n|a_n|^2R^{2n}\end{aligned}$$

and so by monotone convergence, since all the terms of the sum are nonnegative,

$$A(R) = \pi\sum_n n|a_n|^2R^{2n}.$$

By monotone convergence again,

$$|f(\mathbb{D})| = \lim_{R\nearrow 1} A(R) = \pi\lim_{R\nearrow 1}\sum_n n|a_n|^2R^{2n} = \pi\sum_n n|a_n|^2.$$

Exercise 5:

(1) Let $g(z) = z + b_3z^3 + b_5z^5 + \cdots$ and assume that $|b_3| = 1$. Then, if

$$F(z) := \frac{1}{g(1/z)}$$

we have the Laurent expansion $F(z) = z + \frac{-b_3}{z} + \cdots = z + d_0 + d_1z^{-1} + \cdots$. To this function, we can apply Grönwall's Area Theorem:

$$|b_3| + \sum_{n=2}^{\infty}|d_n|^2 \leq 1 \quad\Rightarrow\quad d_n = 0 \quad \forall n\geq 2$$

Hence,

$$z + \frac{-b_3}{z} = F(z) = \frac{1}{g(1/z)}.$$

Rearranging this and letting $z\mapsto 1/z$, we get

$$g(z) = \frac{z}{1-b_3z^3}.$$

(2) Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and assume that $|a_2| = 2$. Then, if

$$F(z) = \Big(f(z^2)\Big)^{1/2}$$

we have the power series expansion $F(z) = z + \frac{a_2}{2}z^3 + \cdots = z + d_1 z^3 + \cdots$. To this odd schlicht function, we can apply the first part of this exercise to obtain that

$$f(z^2) = F(z)^2 = \frac{z^2}{(1 - e^{i\theta}z^2)^2} \quad \Rightarrow \quad f(z) = \frac{z}{(1 - e^{i\theta}z)^2}$$

Exercise 6: First, note that if $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$, then $f'(0) = a_1 \neq 0$, for otherwise f would not be injective. Now,

$$g(z) := \frac{f(z) - a_0}{a_1} = z + \frac{a_2}{a_1}z^2 + \cdots$$

is schlicht, and by Bieberbach's inequality,

$$\left|\frac{a_2}{a_1}\right| \leq 2 \quad \Rightarrow \quad |a_2| \leq 2|a_1|.$$

Since $f'(0) = a_1$ and $f''(0) = 2a_2$, we get

$$|f''(0)| \leq 4|f'(0)|.$$

Exercise 10: This proof can be found in Harmonic Measure by Garnett and Marshall. Let $z_0 \in \mathbb{D}$ be fixed.

(i) Set

$$\psi(z) := \frac{f\left(\frac{z+z_0}{1+\overline{z_0}z}\right) - f(z_0)}{f'(z_0)(1 - |z_0|^2)}$$

This is a normalized univalent function with $\psi(0) = 0$ and $\psi'(0) = 1$, and has a power series expansion

$$\psi(z) = z + a_1 z + a_2 z^2 + \cdots.$$

From a calculation,

$$\psi''(0) = f''(z_0)\frac{1 - |z_0|^2}{f'(z_0)} - 2\overline{z_0}$$

and hence by Rescaled Bieberbach's inequality ,

$$\frac{1}{2}\left|f''(z_0)\frac{1 - |z_0|^2}{f'(z_0)} - 2\overline{z_0}\right| = \left|\frac{\psi''(0)}{2}\right| = |a_2| \leq 2,$$

or equivalently,

$$\left|f''(z_0)\frac{1 - |z_0|^2}{f'(z_0)} - 2\overline{z_0}\right| \leq 4.$$

Multiplying both sides of this inequality by $|z_0|/(1-|z_0|^2)$ gets the desired inequality.

(ii) Set $z_0 = r_0 e^{i\theta_0}$. Then

$$\left| f''(z_0)\frac{1-|z_0|^2}{f'(z_0)} - 2\overline{z_0} \right| = \left| e^{i\theta_0} f''(z_0)\frac{1-|z_0|^2}{f'(z_0)} - 2|z_0| \right| \leq 4$$

and therefore

$$\left| e^{i\theta_0}\frac{f''(z_0)}{f'(z_0)} - \frac{2|z_0|}{1-|z_0|^2} \right| \leq \frac{4}{1-|z_0|^2}$$

Since $\frac{2|z_0|}{1-|z_0|^2}$ is real,

$$\left| \Re\Big(e^{i\theta_0}\frac{f''(z_0)}{f'(z_0)}\Big) - \frac{2|z_0|}{1-|z_0|^2} \right| \leq \left| e^{i\theta_0}\frac{f''(z_0)}{f'(z_0)} - \frac{2|z_0|}{1-|z_0|^2} \right| \leq \frac{4}{1-|z_0|^2}$$

Therefore, noting that $z_0/|z_0| = e^{i\theta_0}$,

$$\frac{2|z_0|-4}{1-|z_0|^2} \leq \Re\Big(\frac{z_0}{|z_0|}\frac{f''(z_0)}{f'(z_0)}\Big) \leq \frac{2|z_0|+4}{1-|z_0|^2}. \tag{1}$$

Let $g = u + iv$ be a holomorphic function. Then, note that if $z = re^{i\theta}$

$$\frac{\partial \Re g}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$

and

$$\begin{aligned}\frac{z}{|z|}g'(z) &= \Big(\cos(\theta) + i\sin(\theta)\Big)\Big(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\Big)\\ &= \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta) + i\Big(\frac{\partial u}{\partial y}\sin(\theta) - \frac{\partial u}{\partial y}\cos(\theta)\Big).\end{aligned}$$

Hence,

$$\Re\left(\frac{z}{|z|}g'(z)\right) = \frac{\partial \Re g}{\partial r}(z).$$

Note that $\mathbb{D}$ is a simply connected domain, and $f'(\mathbb{D}) \not\ni 0$ since f is univalent. Hence $g = \log f'$ is well defined, and $g' = f''/f'$. Hence by (1) we obtain

$$\frac{2|z_0|-4}{1-|z_0|^2} \leq \frac{\partial}{\partial r_0}\log|f'(z_0)| \leq \frac{2|z_0|+4}{1-|z_0|^2}$$

If $z = re^{i\theta_0}$, then

$$\int_0^r \frac{2|z_0|-4}{1-|z_0|^2}dr_0 \leq \int_0^r \frac{\partial}{\partial r_0}\log|f'(z_0)|dr_0 \leq \int_0^r \frac{2|z_0|+4}{1-|z_0|^2}dr_0$$

$$\Downarrow$$

$$\log\frac{1-|z|}{(1+|z|)^3} \leq \log|f'(z)| \leq \log\frac{1+|z|}{(1-|z|)^3}$$

Since the exponential function is increasing, we get (ii).

(iii) From the upper bound in (ii) one gets

$$\ell(\psi([0,z])) = \int_0^r |f'(z_0)|dr_0 \leq \int_0^r \frac{1+|z_0|}{(1-|z_0|)^3}dr_0 = \frac{|z|}{(1-|z|)^2}$$

where ℓ is the path length, and $[0,z]$ is the line from 0 to z. By a trivial estimate, $|f(z)| \leq \ell(f([0,z]))$, and so we get the upper bound in (iii):

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

For the lower bound, we can assume that $|f(z)| < 1/4$, since

$$\frac{|z|}{(1+|z|)^2} < \frac{1}{4}$$

Now, by Koebe's Quarter therorem, $f(\mathbb{D}) \supset D(1/4, 0)$. Hence for $t \in [0, |f(z)|]$ since f is univalent we can define

$$\gamma(t) = f^{-1}(te^{i\arg(z)}) \in \mathbb{D}.$$

Then by the lower bound in (ii), we get

$$\int_\gamma \frac{1-|z|}{(1+|z|)^3}|dz| \leq \int_\gamma |f'(z)||dz| = \int_0^{|f(z)|} dt = |f(z)|$$

Then since $\frac{1-|z|}{(1+|z|)^3}$ depends only on the distance from the origin and is always positive, the straight radial path from 0 to z minimizes the contribution of the integrand and is clearly of length less than γ. Hence

$$\int_\gamma \frac{1-|z|}{(1+|z|)^3}|dz| \geq \int_0^{|z|} \frac{1-t}{(1+t)^3}dt = \frac{|z|}{(1+|z|)^2}$$

which is the lower bound of (iii).

(iv) Now, applying (iii) to $\psi(-z_0)$, we get

$$\frac{|z_0|}{(1+|z_0|)^2} \leq \frac{|f(z_0)|}{|f'(z_0)(1-|z_0|^2)|} \leq \frac{|z_0|}{(1-|z|_0)^2}$$

Rearranging this, we get (iv).

(v) Let $K \subset \mathbb{D}$ be compact. Then, there exists an $0 < R < 1$ such that $K \subset D(0,R) \subset \mathbb{D}$ all strict. Then, by (iii), for all $z \in D(0,R)$ and $n \in \mathbb{N}$

$$|f_n(z)| \leq \frac{R}{(1-R)^2}$$

and f_n is uniformly bounded on K. Since K was arbitrary, this shows that the family $\{f_n\}$ is uniformly bounded on every compact subset of $\mathbb{D}$. By Theorem 15 on page 224 of Ahlfor's Complex Analysis, a family of analytic functions which is uniformly bounded on compact subsets is normal.

(vi) By Theorem 16 on page 225 of Ahlfor's Complex Analysis, a locally bounded family of analytic functions has locally bounded derivates. Since $\{0\}$ is compact, there exists a C_n such that

$$|f^{(n)}(0)| \leq C_n$$

which is what we wanted to show.

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