HOMEWORK 2, MATH246C, SPRING 2018

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Exercise 37: Using the above proposition, show that if X is a compact Riemann surface and P, Q are distinct points in X, then there is a meromorphic 1-form $\omega_{(P)-(Q)}$ on X with poles only at P, Q, with a residue of 1 at P and a residue of -1 at Q.

Using this, conclude the Riemann existence theorem: for any compact Riemann surface X and distinct points P, Q in X, there exists a meromorphic function on X that takes different values at P and Q and is in particular non-constant. (In other words, the meromorphic functions separate points.)

Solution 37: From Proposition 36, there exists a harmonic function $f : X \setminus \{P, Q\} \to \mathbb{R}$ with the property that for coordinate charts (U_α, ϕ_α), (U_β, ϕ_β) such that $P \in U_\alpha$, $Q \in U_\beta$ and $\phi_\alpha(P) = \phi_\beta(Q) = 0$, we have that

$$f \circ \phi_\alpha^{-1}(z) = \log|z| + g(z)$$
$$f \circ \phi_\beta^{-1}(z) = -\log|z| + h(z)$$

where g, h are bounded functions. Since g, h are bounded, they can be extended to be harmonic functions on the whole domain since $\pm\log|z|$ is harmonic on the punctured plane [Ahlfors]. Now, for any coordinate chart (U_γ, ϕ_γ) such that $U_\gamma \not\ni P, Q$, define

$$(\omega_{(P)-(Q)})_{\phi_\gamma} dz := (Df)_{\phi_\gamma}.$$

For $U_\alpha \ni P$, we can without loss of generality assume $\phi_\alpha(P) = 0$. Define

$$(\omega_{(P)-(Q)})_{\phi_\alpha} dz := (D\log|z|)_{\phi_\alpha} dz + (Dg)_{\phi_\alpha} dz$$

and for $U_\beta \ni Q$ with $\phi_\beta(Q) = 0$ define

$$(\omega_{(P)-(Q)})_{\phi_\beta} dz := (-D\log|z|)_{\phi_\beta} dz + (Dh)_{\phi_\beta} dz.$$

It is clear by Exercise 35 that the transition functions satisfy the correct relations for meromorphic 1-forms. Moreover, by a tedious computation

$$(D\log|z|)_{\phi_\alpha} dz = \left(\frac{1}{z}\right)_{\phi_\alpha} dz$$
$$(-D\log|z|)_{\phi_\beta} dz = \left(-\frac{1}{z}\right)_{\phi_\beta} dz$$

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Hence $\omega_{(P)-(Q)}$ has a simple pole of residue 1 at P and a simple pole of residue -1 at Q, and no other poles.

Now, let $P, P', Q, Q' \in X$ be four distinct points. Then, the meromorphic function

$$F := \frac{\omega_{(P)-(Q')}}{\omega_{(P')-(Q)}}$$

Has the property that $F(P) = \infty$, and $F(Q) = 0$. $\square$

Exercise 41: Let X be a compact Riemann surface of genus one, and let ∞ be a point on X. Show that for any points P, Q on X, there is a unique point $P+Q$ on X such that $(P)+(Q)-(P+Q)-(\infty)$ is a principal divisor. Furthermore show that this defines an abelian group law on X. What is this group law in the case that X is an elliptic curve?

Solution 41: The space of holomorphic 1-forms has dimension $g = 1$. Hence, there exists a holomorphic 1-form ω which is non-trivial. Since ω has no poles, it has no zeros. Let $P, Q \in X$ be distinct. Then, by Exercise 37, there exists a meromorphic 1-form $\omega_{(P)-(Q)}$ with poles exactly at P and Q. Now, in a local coordinate chart (U_α, ϕ_α) centered around ∞, we have

$$(\omega_{(P)-(Q)} - c\omega)_\alpha(0) = (\omega_{(P)-(Q)})_\alpha - c(\omega)_\alpha(0).$$

Hence, by picking a $c \in \mathbb{C}$ correctly, $\omega_{(P)-(Q)} - c\omega$ will have a zero at ∞. Because the zeros of a meromorphic 1-form are defined locally, this makes sense. Since $\omega_{(P)-(Q)} - c\omega$ is a meromorphic 1-form, its degree is $2g - 2 = 0$. It has poles at exactly P and Q and a zero at ∞, so there exists precisely one other zero $R \in X$ (which may be ∞). Then, the quotient

$$f := \frac{\omega}{\omega_{(P)-(Q)} - c\omega}$$

is a meromorphic function on X. Because ω has no zeros or poles, f has zeros precisely at P, Q and poles at R, ∞. Hence,

$$(f) = (P) + (Q) - (R) - (\infty)$$

and there exists an R with the required property. If $P = Q$, the above argument works with $\omega_{2(P)}$ instead of $\omega_{(P)-(Q)}$. The existence of $\omega_{2(P)}$ is guaranteed by Proposition 38.

To show it is unique, assume there are two point $R, R' \in X$ such that

$$(f) = (P) + (Q) - (R) - (\infty)$$

$$(g) = (P) + (Q) - (R') - (\infty)$$

Then, the quotient f/g is a meromorphic function, and

$$\left(\frac{f}{g}\right) = (R') - (R)$$

If $R \neq R'$, f/g is a meromorphic function on X with precisely 1 pole. By Exercise 42, f/g is an isomorphism between X and $\mathbb{C}^*$, which is of genus zero. By observing the proof of Exercise 42, this is not a circular argument, and indeed forms a contradiction since X is assumed to be of genus 1. Therefore $R = R'$, and we are justified in defining $P + Q := R$. Note that it is clear from the above construction that the order of the points P, Q does not change the point R, so the proposed abelian structure is commutative.

It remains to check that $(X, +)$ is associative and has inverses. We first check it is associative. Let $P, Q, R \in X$. Then, by definition, there exists $f, g, h, i \in M(X)$ such that

$$\begin{aligned}(f) &= (P) + (Q + P) - (P + (Q + R)) - (\infty)\\ (g) &= (P + Q) + (R) - ((P + Q) + R) - (\infty)\\ (h) &= (Q) + (R) - (Q + R) - (\infty)\\ (i) &= (P) + (Q) - (P + Q) - (\infty).\end{aligned}$$

By a short calculation,

$$\left(\frac{fh}{gi}\right) = ((P + Q) + R) - (P + (Q + R)),$$

and by the same argument given above, we have

$$((P + Q) + R) = (P + (Q + R)).$$

For the existence of inverses, let $P \in X$. Call the point $\infty \in X$ the identity. We want to show that there exists a point $Q \in X$ such that $P + Q = \infty$. Consider the meromorphic 1-form $\omega_{2(\infty)}$. Using the same method as in the proof of the existence of $P + Q$, there is a constant $c \in \mathbb{C}$ such that

$$\Delta := \frac{\omega_{2(\infty)} - c\omega}{\omega}$$

has a double pole at ∞, no other poles, a zero at P, and one zero at some point P'. Hence,

$$(\Delta) = (P) + (P') - (\infty) - (\infty)$$

and $P + P' = \infty$ by uniqueness. This makes $(X, +)$ an abelian group, with ∞ as the identity.

For the case when X is an elliptic curve, we can assume by the converse to Exercise 30 that $X \simeq \mathbb{C}/\Lambda$. Then $\infty := 0 + \Lambda$ and the regular addition operation on $\mathbb{C}$ furnishes the group operation on $\mathbb{C}/\Lambda$.

Exercise 42: Let X be a compact Riemann surface, and there exists a meromorphic function f on X with one simple pole and no other poles. Show that f is an isomorphism between X and the Riemann sphere. Conclude in particular that the Riemann sphere is the only genus zero compact Riemann surface (up to isomorphism, of course).

Solution 42: Assume f is a meromorphic function on X with one simple pole at P and no other poles. Given any point $c \in \mathbb{C}$, the function $f - c$ also has a simple pole at P and no other poles. By Proposition 24, the degree of f is zero, so there must be precisely one point $Q \neq P$ such that $(f - c)(Q) = f(Q) - c = 0$. Hence $f : X \to \mathbb{C}^*$ is one-to-one and onto, and by the definition of meromorphic, it is holomorphic as a map between Riemann surfaces. Hence f is an isomorphism.

Now, let X be a compact Riemann surface of genus 0, and let $P \in X$. Then, any meromorphic 1-form on X has degree -2. By Proposition 38, there exist meromorphic 1-forms $\omega_{n(P)}$ with a pole of order $n \geq 2$ at P, and no other poles. Since $\omega_{2(P)}$ is of degree -2, it has no zeros. Hence, the meromorphic function

$$f := \frac{\omega_{3(P)}}{\omega_{2(P)}}$$

has a pole of order $3 - 2 = 1$ at P, and no other poles. Therefore, X is isomorphic to $\mathbb{C}^*$ by what we have shown above. $\square$

Exercise 43: Let X be a compact Riemann surface of genus g, and let D be a divisor of degree $2g - 2$. Show that $\dim L(D) = g$ when D is a canonical divisor, and $\dim L(D) = g - 1$ otherwise.

Solution 43: Let $D \in \mathrm{Div}(X)$ with $\deg(D) = 2g - 2$. Then, by the Riemann-Roch theorem,

$$\dim L(D) = g - 1 + \dim L(K - D) \tag{1}$$

for any canonical divisor K. Fix K. Since $\deg(D) = \deg(K) = 2g - 2$ for any canonical divisor, $\dim L(K - D) = 0$ or 1 by Exercise 25 (ii), with the latter case occurring if and only if $K - D$ is principal. Now, I claim that $K - D$ is principal if and only if D is a canonical divisor.

For one direction of the claim, assume that $K - D$ is principal. Then there exists an $f \in M(X)$ such that $K - D = (f)$. Let ω be the meromorphic 1-form associated to K, and form the meromorphic 1-form

$$\frac{\omega}{f} =: \omega'$$

Since $(\omega') = (\omega) - (f) = D$, we have that D is a canonical divisor. Now, for the other direction of the claim, assume that D is a canonical divisor with associated meromorphic 1-form ω'. Then

$$f := \frac{\omega}{\omega'}$$

is a meromorphic function with the property that $(f) = (\omega) - (\omega') = K - D$. Therefore, $K - D$ Is principal.

Now, assume that D is a canonical divisor. Then $K - D$ is principal, and hence $\dim L(K - D) = 1$. Hence by (1)

$$\dim L(D) = g - 1 + \dim L(K - D) = g - 1 + 1 = g.$$

Likewise, assume that D is not a canonical divisor. Then $K - D$ is not principal, and $\dim L(K - D) = 0$. Then by (1)

$$\dim L(D) = g - 1 + \dim L(K - D) = g - 1$$

which completes the exercise. □

Exercise 44 (ii), Noether gap theorem: Let X be a compact Riemann surface of genus g.

If $P_1, P_2, \dots$ are a sequence of distinct points in X, show that there are precisely g positive integers n with the property that there does not exist a meromorphic function with a simple pole at P_n, at most a simple pole at $P_1, \dots, P_{n-1}$, and no other poles. Show in addition that all of these integers are less than or equal to $2g - 1$.

Solution 44 (ii): Define $L_0 := L(0)$ and $L_n := L((P_1) + \cdots + (P_n))$. Now, by the Riemann-Roch theorem,

$$\dim L_k = \dim L(K - \sum_{l=1}^{k} (P_k)) + k + 1 - g.$$

Then, since $\deg K = 2g - 2$, for $k \geq 2g - 1$ we have

$$\deg(K - \sum_{l=1}^{k} (P_k)) = 2g - 2 - k \leq -1 < 0.$$

Hence by Corollary 16

$$\dim L((K - \sum_{l=1}^{k} (P_k)) = 0$$

for $k \geq 2g - 1$. Therefore, for $k \geq 2g - 1$,

$$\dim L_k = k + 1 - g.$$

Consider the sequence

$$\dim L_0 = 1, \dim L_1, \dots, \dim L_{2g-1} = g, \dim L_{2g} = g+1, \dots$$

By Lemma 19, $\dim L_{n+1} - \dim L_n \leq 1$, and so by a simple counting argument there exist exactly g numbers $1 \leq n \leq 2g$ such that

$$\dim L_n = \dim L_{n+1} \qquad \Longrightarrow \qquad L_n = L_{n+1}$$

Now, assume there exists an $f \in M(X)$ with the property that $\mathrm{ord}_{P_{n+1}}(f) = -1$ and $\mathrm{ord}_{P_l}(f) \geq -1$ for each $l = 1, \dots, n$. Then, $f \in L_n \setminus L_{n+1} = \emptyset$, so f cannot exist. □

References: John and Akash for advice.