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Exercise 25: Let Q be a Jordan quadrilateral with $|Q| = A$ and a being the shortest Euclidean distance between the b-sides of Q and a being the shortest Euclidian distance between the a-sides of Q. By Proposition 23, for all Borel measurable $\rho : Q \to [0, \infty)$, there exists a curve γ in Q which connects the a-sides of Q such that

$$\Big(\int_\gamma \rho|dz|\Big)^2 \le \mathrm{mod}(Q)\int_Q \rho^2 dxdy.$$

In particular, for $\rho \equiv 1$, there exists a curve γ_0 in Q such that

$$\mathrm{length}(\gamma_0)^2 = \Big(\int_{\gamma_0} |dz|\Big)^2 \le \mathrm{mod}(Q)\int_Q dxdy = \mathrm{mod}(Q)|Q|$$

and since

$$\mathrm{length}(\gamma_0)^2 \le b^2$$

we have

$$\frac{b^2}{|Q|} \le \mathrm{mod}(Q).$$

Since Q was arbitrary, this shows the lower bound for all Q.

Directly from this, the upper bound follows. Let Q be a quadrilateral, and let Q' be the same quadrilateral with vertices cyclically permuted by 1. Then we clearly have $|Q| = |Q'|$, $b' = a$ and $\mathrm{mod}(Q') = \mathrm{mod}(Q)$. Then from above

$$\frac{b'^2}{|Q'|} \le \mathrm{mod}(Q') \qquad \Leftrightarrow \qquad \frac{a^2}{|Q|} \le \frac{1}{\mathrm{mod}(Q)}$$

which is the upper bound. Note that this argument only works if the lower bound holds *for all* Jordan quadrilaterals.

Let's now tackle the equality case. First assume that

$$\mathrm{mod}(Q) = \frac{|Q|}{a^2}$$

and so in particular

(1) $$\mathrm{mod}(Q) \ge \frac{|Q|}{a^2}.$$

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Now, let $\phi : R \to Q$ be the conformal map where R is a rectangle with vertices $0, M, i+M, i$. Then, $\text{mod}(Q) = M = |R|$. By the change of variables formula, since $|\det D\phi(z)| = |\phi'(z)|^2$ we get

$$|Q| = \int_{\phi\circ\phi^{-1}(Q)} dxdy = \int_R |\phi'(z)|^2 dxdy$$

and by Cauchy-Schwarz

$$(2) \quad |Q| = \int_R |\phi'(z)|^2 dxdy \geq \frac{1}{|R|}\Big(\int_R |\phi'(z)|dxdy\Big)^2 = \frac{1}{\text{mod}Q}\Big(\int_R |\phi'(z)|dxdy\Big)^2$$

By (1) and (2) we get

$$(3) \quad \text{mod}(Q) \geq \frac{1}{a^2\text{mod}Q}\Big(\int_R |\phi'(z)|dxdy\Big)^2 \quad \Leftrightarrow \quad \text{mod}(Q) \geq \frac{1}{a}\int_R |\phi'(z)|dxdy.$$

Now, we can rewrite the above integral by Fubini as

$$\frac{1}{a}\int_0^M \int_0^1 |\phi'(x+iy)|dydx = \frac{1}{a}\int_0^M \int_{\phi\circ\alpha_x} |dz|dx$$

where $\alpha_x(t) = x + it$. Now, because the curve $\phi \circ \alpha_x$ is a curve in Q which connects the b-sides of Q,

$$\frac{1}{a}\int_{\phi\circ\alpha_x} |dz| = \frac{\text{length}(\phi\circ\alpha_x)}{a} \geq 1,$$

and so

$$\int_0^M \Big(\frac{1}{a}\int_{\phi\circ\alpha_x} |dz|\Big)dx \geq \int_0^M 1dx = M = \text{mod}(Q).$$

Combining this with (3), we get that

$$\text{mod}(Q) = \frac{1}{a}\int_R |\phi'(z)|dxdy.$$

Now by our assumption (1),

$$\Big(\frac{1}{a}\int_R |\phi'(z)|dxdy\Big)^2 = \text{mod}(Q)^2 \geq \frac{\text{mod}(Q)|Q|}{a^2} = \frac{\text{mod}(Q)}{a^2}\int_R |\phi'(z)|^2 dxdy$$

we get

$$\Big(\int_R |\phi'(z)|dxdy\Big)^2 \geq \text{mod}(Q)\int_R |\phi'(z)|^2 dxdy = |R|\int_R |\phi'(z)|^2 dxdy.$$

This is in fact the opposite inequality we get from Cauchy-Schwarz with the functions χ_R and $|\phi'|$, so by the equality case of Cauchy-Schwarz, $|\phi'| = c\chi_R$ for some $c \in \mathbb{R}$. Therefore $|\phi'|$ is some nonzero constant, and so by the maximum modulus principle ϕ' is constant. Hence ϕ is affine. This shows that $Q = \phi(R)$ is a rectangle since R is, completing one case of the equality.

The other case of the equality is carried out in a similar manner. Let ϕ, R, M be as in the previous part. Then, if

$$\frac{b^2}{|Q|} = \text{mod}(Q)$$

we get in particular that

$$\text{mod}(Q) \leq \frac{b^2}{|Q|}. \tag{4}$$

By the change of variables formula and Cauchy-Schwarz, we have that

$$|Q| = \int_R |\phi'(z)|^2 dxdy \geq \frac{1}{|R|}\Big(\int_R |\phi'(z)| dxdy\Big)^2 = \frac{1}{\text{mod}(Q)}\Big(\int_R |\phi'(z)| dxdy\Big)^2$$

By this and (4), we get that

$$\text{mod}(Q) \leq \frac{\text{mod}(Q) b^2}{\Big(\int_R |\phi'(z)| dxdy\Big)^2} \quad \Leftrightarrow \quad \frac{1}{b}\int_R |\phi'(z)| dxdy \leq 1 \tag{5}$$

As above, we can rewrite this integral as an iterated integral by Fubini:

$$\int_R |\phi'(z)| dxdy = \int_0^1 \int_0^M |\phi'(x+iy)| dxdy = \int_0^1 \int_{\phi\circ\beta_y} |dz| dy$$

where $\beta_y(t) = t + iy$. Note that $\phi \circ \beta_y$ is a curve in Q which connects the two a-sides of Q, and so

$$\frac{1}{b}\int_0^1 \int_{\phi\circ\beta_y} |dz| dy = \frac{\text{length}(\phi\circ\beta_y)}{b} \geq \int_0^1 1 dy = 1.$$

Combining this with (5), we get that

$$\int_R |\phi'(z)| dxdy = b.$$

By our assumption in (4),

$$|R| = \text{mod}(Q) \leq \frac{b^2}{|Q|} = \frac{1}{|Q|}\Big(\int_R |\phi'(z)| dxdy\Big)^2 = \frac{\Big(\int_R |\phi'(z)| dxdy\Big)^2}{\int_R |\phi'(z)|^2 dxdy}.$$

Equivalently,

$$|R|\int_R |\phi'(z)|^2 dxdy \leq \Big(\int_R |\phi'(z)| dxdy\Big)^2$$

which is the opposite inequality of Cauchy-Schwarz with the functions χ_R and $|\phi'|$. Hence, we get equality in Cauchy-Schwarz, and $|\phi'| = c\chi_R$ for some $c \in \mathbb{R}$. Hence $|\phi'|$ is constant, and by the maximum modulus theorem, ϕ' is constant. Thus ϕ is affine, and $Q = \phi(R)$ is a rectangle. This completes the exercise.

Exercise 26: Let $\phi : Q \to R$ be the conformal map which takes Q to the rectangle R with vertices at $0, M, i+M, i$. Note that by Exercise 21, this extends to a vertex preserving map from $\overline{Q} \to \overline{R}$. Then, by continuity of ϕ, the a-sides of $\phi(Q_n)$ converge in the Hausdorff sense to the respective a-sides of $\phi(Q)$, and likewise for b-sides and the vertices of $\phi(Q_n)$.

To show this fact, let $\epsilon > 0$, and pick L to be a particular a or b-side of Q. Pick L_n to be the side of Q_n which corresponds to L in the limit. Then, since L is compact, there exists a uniform $\delta > 0$ such that for each $w \in L$,

$$|z-w| < \delta \quad \Rightarrow \quad |\phi(z) - \phi(w)| < \epsilon.$$

Then by Hausdorff convergence, there exists a large N such that for all $n \geq N$,

$$L^\delta := \bigcup_{w \in L} \{z : |z-w| < \delta\} \supset L_n.$$

Hence, for all $n \geq N$,

$$\phi(L_n) \subset \phi(L)^\epsilon.$$

Using the compactness of L_n, for each $z \in L_n$, there exists a δ_n such that

$$|z-w| < \delta_n \quad \Rightarrow \quad |\phi(z) - \phi(w)| < \epsilon.$$

Then by Hausdorff convergence again, there exists a large N' such that for all $n \geq N'$

$$(L_n)^{\delta_n} \supset L.$$

Then, we have for all $n \geq N'$,

$$\phi(L) \subset \phi(L_n)^\epsilon$$

Therefore, for $n \geq \max(N, N')$, we have that

$$d_H(\phi(L), \phi(L_n)) < \epsilon$$

and the sides of $\phi(L_n)$ converge in the Hausdorff sense to $\phi(L)$. The convergence of the vertices of $\phi(Q_n)$ follows from continuity of ϕ.

Therefore, since the modulus is conformally invariant, it suffices to show the claim for Q a rectangle. So, assume that Q is a rectangle. Now, from Exercise 25, we have that

$$\frac{b_n^2}{|Q|} \leq \frac{b_n^2}{|Q_n|} \leq \operatorname{mod}(Q_n) \leq \frac{|Q_n|}{a_n^2} \leq \frac{|Q|}{a_n^2}.$$

Since $|Q| = M = \operatorname{mod}(Q)$, we have that

$$\frac{b_n^2}{\operatorname{mod}(Q)} \leq \operatorname{mod}(Q_n) \leq \frac{\operatorname{mod}(Q)}{a_n^2}.$$

Let $\epsilon > 0$. Pick N so large that each of the sides and vertices of Q_n lie within ϵ Hausdorff distance from the corresponding sides of Q for $n \geq N$. Then, pick the α_n, α_n' on opposing b-sides of Q_n such that $|\alpha_n - \alpha_n'| = a_n$. Assume without loss of generality that $\Im(\alpha_n') < \Im(\alpha_n)$. Then, $\Im(\alpha_n) > 1-\epsilon$ and $\Im(\alpha_n') < \epsilon$ by the Hausdorff distance condition, and so

$$a_n = |\alpha_n - \alpha_n'| \geq \Im(\alpha_n - \alpha_n') > 1 - 2\epsilon.$$

Likewise, let β_n and β_n' be two points on the opposing a-sides of Q_n such that $|\beta_n - \beta_n'| = b_n$. Then assuming without loss of generality that $\Re(\beta_n') < \Re(\beta_n)$, we have by the Hausdorff distance condition that $\Re(\beta_n') < \epsilon$ and $\Re(\beta_n) > M - \epsilon$. Hence

$$b_n = |\beta_n - \beta_n'| \geq \Re(\beta_n - \beta_n') \geq M - 2\epsilon$$

Combining these lower bounds together, we get that

$$\frac{(\mathrm{mod}(Q) - 2\epsilon)^2}{\mathrm{mod}(Q)} = \frac{(M-2\epsilon)^2}{\mathrm{mod}(Q)} \leq \mathrm{mod}(Q_n) \leq \frac{\mathrm{mod}(Q)}{(1-2\epsilon)^2}.$$

Therefore, taking ϵ to zero shows that $\mathrm{mod}(Q_n) \to \mathrm{mod}(Q)$.

Exercise 32: Let $Q \subset U$ be a Jordan rectangle. By pre-composing and post-composing ϕ with suitable conformal mappings, we may assume without loss of generality that both Q and $\phi(Q)$ are normalized rectangles (holomorphic maps send sets of measure zero to sets of measure zero). Assume that Q has vertices $0, M, i+M, i$, and that $\phi(Q)$ has vertices $0, M', i+M', i$. We claim that $M' \leq M$.

To show this, we emulate the proof of Theorem 29. First, note that by the bijectivity of ϕ, $\phi(Q \setminus S) = \phi(Q) \setminus \phi(S)$. Then, on $Q \setminus S$, ϕ is conformal, and so the change of variables formula applies;

$$|\phi(Q)| = \int_{\phi(Q)} dxdy = \int_{\phi(Q\setminus S)} dxdy + \int_{\phi(S)} dxdy = \int_{Q\setminus S} |\det D\phi(z)| dxdy + \int_{\phi(S)} dxdy$$

which implies that

$$|\phi(Q)| \geq \int_{Q\setminus S} |\det D\phi(z)| dxdy.$$

By the singular value decomposition,

$$|\det D\phi(z)| = \max_{w\in\mathbb{S}^1} |\nabla_w \phi(z)| \min_{v\in\mathbb{S}^1} |\nabla_v \phi(z)| \geq \max_{w\in\mathbb{S}^1} |\nabla_w \phi(z)| \Big|\frac{\partial\phi}{\partial x}(z)\Big|$$

Since ϕ is 1-conformal on $Q \setminus S$, by Theorem 29 we have

$$\max_{w\in\mathbb{S}^1} |\nabla_w \phi(z)| \geq \Big|\frac{\partial\phi}{\partial x}(z)\Big|$$

which implies

$$|\det D\phi(z)| \geq \Big|\frac{\partial\phi}{\partial x}(z)\Big|^2.$$

By Fubini's theorem and Cauchy-Schwarz,

$$|\phi(Q)| \geq \int_{Q\setminus S} \Big|\frac{\partial \phi}{\partial x}(z)\Big|^2 dxdy = \int_0^1 \int_0^M \Big|\frac{\partial \phi}{\partial x}(z)\Big|^2 dxdy \geq \int_0^1 \frac{1}{M}\Big(\int_0^M \Big|\frac{\partial \phi}{\partial x}(z)\Big|dx\Big)^2 dy.$$

By the triangle inequality,

$$\int_0^1 \frac{1}{M}\Big(\int_0^M \Big|\frac{\partial \phi}{\partial x}(z)\Big|dx\Big)^2 dy \geq \int_0^1 \frac{1}{M}\Big|\int_0^M \frac{\partial \phi}{\partial x}(z)dx\Big|^2 dy.$$

Since ϕ is K-quasiconformal, by Proposition 31, ϕ is absolutely continuous on a.e. horizontal line $[0, M] \times \{y\}$. Hence, it satisfies the fundamental theorem of calculus (see e.g. Wheeden and Zygmund's Measure and Integral, Chapter 7) and we have

$$\int_0^1 \frac{1}{M}\Big|\int_0^M \frac{\partial \phi}{\partial x}(z)dx\Big|^2 dy = \int_0^1 \frac{1}{M}|\phi(M+iy) - \phi(iy)|^2 dy \geq \int_0^1 \frac{1}{M}(M')^2 dy.$$

Therefore,

$$|\phi(Q)| \geq \frac{(M')^2}{M}$$

and since $|\phi(Q)| = M'$,

$$M \geq M'$$

which is what we wanted to show.

Exercise 35:

(i) Let's first show that $M = \text{mod}(A) \leq \log(R/r)$. By definition, if $\rho(z) := |z|^{-1}$, there exist a γ winding once around the "hole" of A such that

$$\Big(\int_\gamma \rho|dz|\Big)^2 \leq \frac{2\pi}{M}\int_A \rho^2 dxdy.$$

Now, we have by the residue theorem that

$$\int_\gamma \rho|dz| = \int_\gamma \frac{1}{|z|}|dz| \geq \Big|\int_\gamma \frac{1}{z}dz\Big| = |2\pi i|$$

and moreover by a direct computation,

$$\begin{aligned}\int_A \rho^2 dxdy &= \int_A \frac{1}{|z|^2}dxdy \\ &= \int_0^{2\pi}\int_r^R \frac{1}{t^2} t\,dt\,d\theta \\ &= 2\pi \log\big(\frac{R}{r}\big).\end{aligned}$$

Combining these two estimates implies

$$(2\pi)^2 \leq \frac{2\pi}{M} 2\pi \log\big(\frac{R}{r}\big) \quad \Rightarrow \quad M \leq \log\big(\frac{R}{r}\big).$$

For the lower bound, we have to show that for a fixed $\rho \in L^2(A)$, there exists a γ winding once around $r\mathbb{D}$ such that

$$\Big(\int_\gamma \rho|dz|\Big)^2 \leq \frac{2\pi}{\log(R/r)} \int_A \rho^2 dxdy.$$

To show this, let $\epsilon > 0$ and define

$$\gamma_t(\theta) := te^{i\theta}$$

Let α correspond to the γ_t which satisfies

$$\int_\alpha \rho|dz| \leq \int_{\gamma_t} \rho|dz| + \epsilon$$

for all $r < t < R$. Then, we have that for any $r < t < R$,

$$\int_{\gamma_t} \rho|dz| = \int_0^{2\pi} \rho(te^{i\theta})tdt \quad \Rightarrow \quad \frac{1}{t}\int_\alpha \rho|dz| \leq \int_0^{2\pi} \rho(te^{i\theta})d\theta + \frac{\epsilon}{t}$$

Integrating this inequality over t from r to R we get

$$\log(R/r)\int_\alpha \rho|dz| \leq \int_r^R \int_0^{2\pi} \rho(te^{i\theta})d\theta dt. + \log(R/r)\epsilon$$

By Cauchy-Schwarz,

$$\begin{aligned}\int_r^R \int_0^{2\pi} \rho(te^{i\theta})d\theta dt &\leq \Big(\int_r^R \int_0^{2\pi} \rho^2(te^{i\theta})td\theta dt\Big)^{1/2}\Big(\int_r^R \int_0^{2\pi} \frac{1}{t}d\theta dt\Big)^{1/2} \\ &= \sqrt{2\pi}\log(R/r)^{1/2} \int_r^R \int_0^{2\pi} \rho^2(te^{i\theta})tdtd\theta \\ &= \sqrt{2\pi}\log(R/r)^{1/2} \int_A \rho^2 dxdy.\end{aligned}$$

Combining these inequalities, we get

$$\log(R/r)\int_\alpha \rho|dz| \leq \sqrt{2\pi}\log(R/r)^{1/2}\Big(\int_A \rho^2 dxdy\Big)^{1/2} + \epsilon\log(R/r)$$

and so

$$\int_\alpha \rho|dz| \leq \frac{\sqrt{2\pi}}{\log(R/r)^{1/2}}\Big(\int_A \rho^2 dxdy\Big)^{1/2} + \epsilon.$$

Squaring this, we see that

$$\Big(\int_\alpha \rho|dz|\Big)^2 \leq \frac{2\pi}{\log(R/r)}\Big(\int_A \rho^2 dxdy\Big) + o(1).$$

Hence, we see that $\log(R/r)$ satisfies the inequality, and $M = \log(R/r)$.

(ii) Let C_1 be the inner Joran curve and C_2 the outer Jordan curve. Let L denote a straight line connecting C_1 and C_2 which is contained in A. This curve exists by considering the shortest distance between two points on C_1 and C_2. By the compactness of C_1, C_2 there exist points $z_1 \in C_1$ and $z_2 \in C_2$ which attain the minimum. The line connecting these two point must be contained in A, for otherwise the line would intersect C_1 or C_2 at a point which reduces the distance.

Now, by the openness of A and the compactness of L, There exists an $\epsilon > 0$ such that $L^\epsilon \subset A$. Let $\epsilon > \eta > 0$ and consider the Jordan curve A_η formed by removing the strip L^η from A and parametrizing the boundary. The vertices of A_η are chosen such that the a-sides of A_η are the lines corresponding to L^η. We will show that

$$\operatorname{mod}(A_\eta) \to \frac{2\pi}{\operatorname{mod}(A)} \quad \text{as } \eta \to 0.$$

To show that, it suffices to establish

$$\operatorname{mod}(A_\eta) \le \frac{2\pi}{\operatorname{mod}(A)} \tag{6}$$

$$\frac{2\pi}{\operatorname{mod}(A)} - o(1) \le \operatorname{mod}(A_\eta) \tag{7}$$

Let's first establish the easier inequality (6). Let $\rho \in L^2(A_\eta)$ be arbitrary. Then, we can define $\tilde{\rho}$ to be equal to ρ on A_η, and equal to 0 on the complement. Then, by the definition of modulus for A, there exists a $\tilde{\gamma}$ such that

$$\Big(\int_{\tilde{\gamma}} \tilde{\rho}|dz|\Big)^2 \le \frac{2\pi}{\operatorname{mod}(A)} \int_A \tilde{\rho}^2 dxdy$$

Let γ be the restriction of $\tilde{\gamma}$ to A_η. Then

$$\Big(\int_{\gamma} \rho|dz|\Big)^2 = \Big(\int_{\tilde{\gamma}} \tilde{\rho}|dz|\Big)^2$$

$$\int_{A_\eta} \rho^2 dxdy = \int_A \tilde{\rho}^2 dxdy$$

and so

$$\Big(\int_{\gamma} \rho|dz|\Big)^2 \le \frac{2\pi}{\operatorname{mod}(A)} \int_{A_\eta} \rho^2 dxdy.$$

Since $\operatorname{mod}(A_\eta)$ is defined to be the smallest constant such that this inequality holds for all ρ, we have that

$$\operatorname{mod}(A_\eta) \le \frac{2\pi}{\operatorname{mod}(A)}$$

Now, we prove the much harder direction (7). For any $\eta > 0$, there exist $\rho \in L^2(A)$ such that for all $\gamma \subset A$ curves,

$$\Big(\int_\gamma \rho|dz|\Big)^2 > \Big(\frac{2\pi}{\operatorname{mod}(A)} - \eta\Big)\int_A \rho^2 dxdy.$$

By the scaling invariance of this inequality, we can normalize to assume that $\|\rho\|_2 = 1$ so that

$$\Big(\int_\gamma \rho|dz|\Big)^2 > \frac{2\pi}{\operatorname{mod}(A)} - \eta$$

Note that by Proposition 23, $\rho|_{A_\eta} \in L^2(A_\eta)$, there exists an $\alpha \subset A_\eta$ such that

$$\begin{aligned}\Big(\int_\alpha \rho|_{A_\eta}|dz|\Big)^2 &\leq \operatorname{mod}(A_\eta)\int_{A_\eta} \rho|_{A_\eta}^2 dxdy\\ &\leq \operatorname{mod}(A_\eta)\int_A \rho^2 dxdy\\ &= \operatorname{mod}(A_\eta).\end{aligned}$$

Now, our goal is to show that there exists a path γ such that

$$\int_\gamma \rho|dz| \leq \int_\alpha \rho|_{A_\eta}|dz| + o(1),$$

for combining the two inequalities above would get (7). To this end, note that on L^η, we can integrate along parallel paths to L and use that

$$\int_{L^\eta} \rho \leq |L^\eta|^{1/2}\|\rho\|_2 = |L^\eta|^{1/2}$$

to show that there exists one parallel path β to L contained in L^η such that

$$\int_\beta \rho|dz| \leq |L^\eta|^{1/2} = o(1).$$

Now, at one endpoint of α, continue α in the region L^η in a straight line perpendicular to L until it reaches β. Likewise for the other endpoint of α. Call these paths σ_1, σ_2. Let γ be the path (with the obvious orientations) $\alpha + \sigma_1 + \beta' + \sigma_2$, where β' is the restriction of β to the points where σ_1 and σ_2 meet it. Then, since the lengths of σ_k are bounded in length by 2η, we have that

$$\int_{\sigma_k} \rho|dz| = o(1).$$

Hence,

$$\int_\gamma \rho|dz| = \int_\alpha \rho|dz| + o(1).$$

This shows that

$$\frac{2\pi}{\mathrm{mod}(A)} - o(1) \leq \mathrm{mod}(A_\eta)$$

and hence the limit.

To complete the proof, note that if ϕ is a K-quasiconformal map,

$$\mathrm{mod}(\phi(A)) = \lim_{\epsilon\to 0} \frac{2\pi}{\mathrm{mod}(\phi(A_\epsilon))} \leq \lim_{\epsilon\to 0} \frac{2\pi K}{\mathrm{mod}(A_\epsilon)} = K\mathrm{mod}(A).$$

(iii) This follows almost trivially from the definitions. Let $\rho \in L^2(A_2)$. Then $\rho|_{A_1} \in L^2(A_1)$, and there exists a γ in A_1 such that

$$\left(\int_\gamma \rho|_{A_1}|dz|\right)^2 \leq \frac{2\pi}{\mathrm{mod}(A_1)} \int_{A_1} \rho|_{A_1}^2 dxdy$$

Since

$$\int_\gamma \rho|_{A_1}|dz| = \int_\gamma \rho|dz|$$

we have

$$\left(\int_\gamma \rho|dz|\right)^2 \leq \frac{2\pi}{\mathrm{mod}(A_1)} \int_{A_1} \rho|_{A_1}^2 dxdy \leq \frac{2\pi}{\mathrm{mod}(A_1)} \int_{A_2} \rho^2 dxdy$$

Since $\mathrm{mod}(A_2)$ is the largest constant for which the above inequality holds, we get that $\mathrm{mod}(A_1) \leq \mathrm{mod}(A_2)$. This completes the proof.