Teaching in Math - M398T Handout Solutions

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Warm up: Calculate following integrals:

$$\int_0^1 xe^x dx \quad \text{and} \quad \iiint_{B(0,1)} |x|^4 dV$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^3$ centered at 0.

Solution: Use integration by parts for the first integral:

$$u = x \quad \Rightarrow \quad du = dx$$

$$dv = e^x dx \quad \Rightarrow \quad v = e^x$$

$$\int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx = 1e^1 - 0e^0 - (e^1 - e^0) = 1$$

$$\boxed{\int_0^1 xe^x dx = 1}$$

For the second integral, use integration by spherical shells:

$$\begin{aligned}
\iiint_{B(0,1)} |x|^4 dx &= \int_0^1 \iint_{\partial B(0,r)} |x|^4 dS(x) dr \\
&= \int_0^1 \iint_{\partial B(0,r)} r^4 dS(x) dr \\
&= \int_0^1 r^4 \text{Area}(\partial B(0,r)) dr \\
&= \int_0^1 r^4 (4\pi r^2) dr \\
&= 4\pi \int_0^1 r^6 dr = \boxed{\frac{4\pi}{7}}
\end{aligned}$$

Problem: First, define $\varphi : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\varphi(x_1, x_2, x_3) := 1 - (x_1^2 + x_2^2 + x_3^2)$$

$$\vec{v}(x_1, x_2, x_3) := \left(x_1^3 + 2x_1x_2^2 + 2x_1x_3^2\,,\ x_2^3 + 2x_2x_3^2\,,\ x_3^3\right)$$

Calculate the integral

$$\iiint_{B(0,1)} \varphi \ \mathrm{div}\vec{v}\ dV$$

where $B(0,1)$ is the unit sphere centered 0 in $\mathbb{R}^3$.

Hint: what is $(x_1^2 + x_2^2 + x_3^2)^2$?

Solution: First use the integration by parts:

$$\iiint_{B(0,1)} \varphi \ \mathrm{div}(\vec{v})dx = \iint_{\partial B(0,1)} \varphi\vec{v} \cdot \vec{n}dS - \iiint_{B(0,1)} \vec{v} \cdot \nabla\varphi dx.$$

Since $\varphi(x) = 1 - 1^2 = 0$ if $|x| = 1$,

$$\iint_{\partial B(0,1)} \varphi\vec{v} \cdot \vec{n}dS = 0.$$

Now by a calculation,

$$\nabla\varphi(x_1, x_2, x_3) = (-2x_1, -2x_2, -3x_3) = -2(x_1, x_2, x_3)$$

and hence

$$\begin{aligned}
\iiint_{B(0,1)} \vec{v} \cdot \nabla\varphi dx &= -2\iiint_{B(0,1)} \left(x_1^3 + 2x_1x_2^2 + 2x_1x_3^2\,,\ x_2^3 + 2x_2x_3^2\,,\ x_3^3\right) \cdot (x_1, x_2, x_3)dx \\
&= -2\iiint_{B(0,1)} \left(x_1^4 + 2x_1^2x_2^2 + 2x_1^2x_3^2 + x_2^4 + 2x_2^2x_3^2 + x_3^4\right)dx \\
&= -2\iiint_{B(0,1)} \left(x_1^2 + x_2^2 + x_3^2\right)^2 dx \\
&= -2\iiint_{B(0,1)} |x|^4 dx = \frac{-8\pi}{7}
\end{aligned}$$

Hence

$$\begin{aligned}
\iiint_{B(0,1)} \varphi \ \mathrm{div}(\vec{v})dx &= \iint_{\partial B(0,1)} \varphi\vec{v} \cdot \vec{n}dS - \iiint_{B(0,1)} \vec{v} \cdot \nabla\varphi dx \\
&= 0 - \frac{-8\pi}{7} = \boxed{\frac{8\pi}{7}}
\end{aligned}$$

Formulas

$$\int u dv = uv - \int v du$$

$$\int_{B(0,R)} f dx = \int_0^R \iint_{\partial B(0,r)} f dS dr$$

$$\iiint_\Omega \varphi \ \mathrm{div}(\vec{v}) dx = \iint_{\partial\Omega} \varphi \vec{v} \cdot \vec{n} dS - \iiint_\Omega \vec{v} \cdot \nabla\varphi dx$$