HEISENBERG'S INEQUALITY AND THE MALGRANGE-EHRENPREIS THEOREM

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It is a well known heuristic that a highly localized function in the spacial domain gives a highly dispersed Fourier transform in the frequency domain. The classical Heisenberg's inequality gives an explicit L^2 or probabilistic description of this phenomenon. Taking these non-localization ideas further, Amrein and Berthier obtained an improved bound with functions with finite support, and Logvinenko and Sereda gave a characterization of "thick" and "thin" sets in phase-space. The Logvinenko-Sereda theorem in particular gives a streamlined proof of the Malgrange-Ehrenpreis theorem for distributional solutions of constant coefficient PDE.

These notes follow [1] closely. To prove these theorems, we first prove the following nice property of L^2 functions with compact support.

Theorem 1: [Bernstein's Bound] Let $f \in L^2(\mathbb{R}^d)$ with $\text{supp}(f) \subset B(0,r)$. Then $\widehat{f} \in \mathcal{C}^\infty(\mathbb{R}^d)$ and we have

$$\|\partial^\alpha \widehat{f}\|_2 \leq (2\pi r)^{|\alpha|}\|f\|_2.$$

Proof 1: Let $k \in \{1, \dots, d\}$ and pick a sequence $h_n \to 0$. Then by dominated convergence

$$\begin{aligned}\lim_{n\to 0} \frac{\widehat{f}(\xi + h_n\xi_k) - \widehat{f}(\xi)}{h_n} &= \lim_{n\to 0} \int_{\mathbb{R}^d} \frac{e^{-2\pi i x_k h_n} - 1}{h_n} f(x) e^{-2\pi i \xi\cdot x} dx \\ &= \int_{\mathbb{R}^d} (-2\pi i x_k) f(x) e^{-2\pi i \xi \cdot x} dx \\ &= (-2\pi i)\widehat{x_k f}(\xi).\end{aligned}$$

It is clear this may be continued for the multiindex α to get

$$\partial^\alpha \widehat{f}(\xi) = (-2\pi i)^{|\alpha|}\widehat{x^\alpha f}(\xi).$$

Hence by Plancherel's Theorem

$$\|\partial^\alpha \widehat{f}\|_2 = (2\pi)^{|\alpha|}\|\widehat{x^\alpha f}\|_2 = (2\pi)^{|\alpha|}\|x^\alpha f\|_2 \leq (2\pi r)^{|\alpha|}\|f\|_2. \quad \square$$

Date: Fall, 2018.

Now, we prove the classic Heisenberg's inequality. It's proof is basically just the observation that the commutator $[D, X]$ of the momentum and position operators is a multiple of the identity.

<u>Theorem 2:</u> [**Heisenberg's Inequality, '27**] For any $f \in \mathcal{S}(\mathbb{R})$ and any $x_0, \xi_0 \in \mathbb{R}$ we have

$$\|f\|_2^2 \leq 4\pi \|(x - x_0) f\|_2 \|(\xi - \xi_0)\widehat{f}\|_2.$$

<u>Proof 2:</u> By considering $g(x) := e^{2\pi i \xi_0 x} f(x + x_0)$ we may take $x_0 = \xi_0 = 0$. Define $X, D : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by

$$Df(x) = \frac{1}{2\pi i} f'(x), \quad \text{and} \quad Xf(x) = xf(x).$$

Then by a computation $[D, X] = \frac{1}{2\pi i} I$. View $\mathcal{S}(\mathbb{R})$ as an inner product space with the standard $L^2(\mathbb{R})$ pairing, and note that X, D are self adjoint. Then we have

$$\begin{aligned}
\|f\|_2^2 &= 2\pi i \langle [D, X] f, f \rangle \\
&= 2\pi i (\langle DXf, f\rangle - \langle XDf, f\rangle). \\
&= 2\pi i (\langle Xf, Df\rangle - \langle Df, Xf\rangle) \\
&= 4\pi \Im \langle Df, Xf\rangle.
\end{aligned}$$

By Cauchy-Schwarz, this is dominated by

$$4\pi \|Df\|_2 \|Xf\|_2$$

and by Plancherel's Theorem we know this is just

$$4\pi \|\widehat{Df}\|_2 \|Xf\|_2.$$

Since $\widehat{Df}(\xi) = \xi \widehat{f}(\xi)$ we are done. $\square$

Theorem 2 gives a "probabilistic" description of the non-locality of a function and its Fourier transform in phase space. A question along a similar vein may be posed as follows: Given measurable, finite sets $E, F \subset \mathbb{R}^d$, does there exist a function $f \in L^2(\mathbb{R}^d)$ such that $\text{supp}(f) \subset E$ and $\text{supp}(\widehat{f}) \subset F$? The following theorem answers this question in the negative.

<u>Theorem 3:</u> [**Amrein-Berthier, '77**] Let $E, F \subset \mathbb{R}^d$ be measurable and have finite measure. Then there exists a constant $C = C(E, F, d)$ such that

$$\|f\|_2 \leq C(\|f\|_{L^2(E^c)} + \|\widehat{f}\|_{L^2(F^c)})$$

for all $f \in L^2(\mathbb{R}^d)$.

To prove this, we need a few lemma. The first one simplifies our problem to showing an operator has norm less than 1.

Lemma 4: Let $E, F \subset \mathbb{R}^d$ be measurable and have finite measure. If there exists a $\rho \in (0,1)$ such that $\|\mathbb{1}_E(\mathbb{1}_F \widehat{f})^\vee\|_2 \le \rho \|f\|_2$ for all $f \in L^2(\mathbb{R}^d)$ then Theorem 3 is true.

Proof 4: Calculate using Plancherel's Theorem that

$$\|f\|^2_{L^2(E^c)} + \|\widehat{f}\|^2_{L^2(F^c)} = \|f\|_2^2 - \|f\|^2_{L^2(E)} + \|\widehat{f}\|_2^2 - \|\widehat{f}\|_{L^2(F)} \tag{1}$$

$$= 2\|f\|_2^2 - (\|\mathbb{1}_E f\|_2^2 + \|(\mathbb{1}_F \widehat{f})^\vee\|_2^2). \tag{2}$$

Assuming (ii) holds, the following holds for any $g \in L^2(\mathbb{R}^d)$ with $\text{supp}(g) \subset E$:

$$\begin{aligned} \|g\|_2 = \|\mathbb{1}_E g\|_2 &= \|\mathbb{1}_E(\mathbb{1}_F \widehat{f})^\vee + \mathbb{1}_E(\mathbb{1}_{F^c}\widehat{g})^\vee\|_2 \\ &\le \|\mathbb{1}_E(\mathbb{1}_F \widehat{g})^\vee\|_2 + \|\mathbb{1}_E(\mathbb{1}_{F^c}\widehat{g})^\vee\|_2 \\ &\le \rho\|g\|_2 + \|(\mathbb{1}_{F^c}\widehat{g})^\vee\|_2 \\ &\le \rho\|g\|_2 + \|\widehat{g}\|_{L^2(F^c)} \end{aligned}$$

In particular, we have

$$g \in L^2(\mathbb{R}^d), \quad \text{and} \quad \text{supp}(g) \subset E \quad \Rightarrow \quad \|g\|_2 \le (1-\rho)^{-1}\|\widehat{g}\|_{L^2(F^c)}. \tag{3}$$

Similarly, if $g \in L^2(\mathbb{R}^d)$ and $\text{supp}(\widehat{g}) \subset F$ we have

$$\begin{aligned} \|g\|_2 = \|\mathbb{1}_F \widehat{g}\|_2 &= \|\mathbb{1}_E(\mathbb{1}_F \widehat{g})^\vee + \mathbb{1}_{E^c}(\mathbb{1}_F\widehat{g})^\vee\|_2 \\ &\le \|\mathbb{1}_E(\mathbb{1}_F \widehat{g})^\vee\|_2 + \|\mathbb{1}_{E^c}(\mathbb{1}_F\widehat{g})^\vee\|_2 \\ &\le \rho\|g\|_2 + \|(\mathbb{1}_F\widehat{g})^\vee\|_{L^2(E^c)} \\ &= \rho\|g\|_2 + \|g\|_{L^2(E^c)} \end{aligned}$$

which implies

$$g \in L^2(\mathbb{R}^d), \quad \text{and} \quad \text{supp}(\widehat{g}) \subset F \quad \Rightarrow \quad \|g\|_2 \le (1-\rho)^{-1}\|g\|_{L^2(E^c)}. \tag{4}$$

Set $C = (1-\rho)^{-1}$. Using (3) and (4), we have that (2) is bounded below by

$$2\|f\|_2^2 - C^2(\|\widehat{f}\|^2_{L^2(F^c)} + \|f\|^2_{L^2(E^c)}).$$

Rearranging this inequality, we have

$$\begin{aligned} \|f\|_2 &\le \sqrt{\frac{C^2+1}{2}}\sqrt{\|\widehat{f}\|^2_{L^2(F^c)} + \|f\|^2_{L^2(E^c)}} \\ &\le \sqrt{C^2+1}(\|\widehat{f}\|_{L^2(F^c)} + \|f\|_{L^2(E^c)}) \end{aligned}$$

This proves the lemma. □

This second lemma shows the eigenspaces of our operator above are finite dimensional.

Lemma 5: Let $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a Hilbert-Schmidt operator with kernel $K \in L^2(\mathbb{R}^{2d})$. That is,

$$Tf(x) := \int_{\mathbb{R}^d} K(x, y)f(y)dy.$$

Then for any $\lambda \in \mathbb{R}$ we have

$$\dim(\{f \in L^2(\mathbb{R}^d) : Tf = \lambda f\}) \leq \lambda^{-2}\|K\|^2_{L^2(\mathbb{R}^{2d})}. \tag{5}$$

Proof 5: Assume $\lambda \neq 0$, and let $\{f_j\}_{j=1}^m$ be an orthonormal subset of the eigenspace above. Then by a computation

$$1 = \langle f_j, f_j\rangle = \frac{1}{\lambda}\langle T_j, f_j\rangle = \int_{\mathbb{R}^{2d}} K(x, y)f_j(x)\overline{f_j}(y)dydx.$$

In particular,

$$\lambda^2 = \left|\int_{\mathbb{R}^{2d}} K(x, y)f_j(x)\overline{f_j}(y)dydx\right|^2.$$

Since $\{f_j \otimes \overline{f_j}\}_{j=1}^m \subset L^2(\mathbb{R}^{2d})$ is also orthonormal, by Bessel's inequality we have

$$m\lambda^2 = \sum_{j=1}^m \left|\int_{\mathbb{R}^{2d}} K(x, y)f_j(x)\overline{f_j}(y)dydx\right|^2 \leq \|K\|^2_{L^2(\mathbb{R}^{2d})}$$

This proves the inequality. □

With Lemma 4 and 5, we may now easily prove Theorem 3. The main idea to proceed by contradiction: If our operator has norm exactly 1, then by a compactness argument and using that translation in the spacial world does not change the support of functions in the phase world, we may construct a sequence of linearly independent eigenfunctions.

Proof 3: Fix E, F finite measure sets, and define $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $Tf := \mathbb{1}_E(\mathbb{1}_F\widehat{f})^\vee$. Note that we may rewrite T as

$$Tf(x) = \mathbb{1}_E(x)((\mathbb{1}_F)^\vee * f(x)) = \int_{\mathbb{R}^d} \mathbb{1}_E(x)\mathbb{1}_F^\vee(x - y)f(y)dy.$$

Setting $K(x, y) := \mathbb{1}_E(x)(\mathbb{1}_F)^\vee(x - y) \in L^2(\mathbb{R}^{2d})$, we see that T is an integral operator with Hilbert-Schmidt kernel K and is therefore a compact operator. Let

$\sigma := \|K\|_{L^2(\mathbb{R}^{2d})} = (|E||F|)^{1/2}$. In light of Lemma 4, our goal is to show that $\|T\|_{2\to 2} < 1$. Since by Plancherel's Theorem

$$\|Tf\|_2 = \|\mathbb{1}_E(\mathbb{1}_F\widehat{f})\|_2 \leq \|(\mathbb{1}_F\widehat{f})^\vee\|_2 = \|\mathbb{1}_F\widehat{f}\|_2 \leq \|\widehat{f}\|_2 = \|f\|_2$$

we have that $\|T\|_{2\to 2} \leq 1$. Assume for the sake of contradiction that $\|T\|_{2\to 2} = 1$. Then there exists a sequence of f_n lying in the unit sphere of $L^2(\mathbb{R}^d)$ such that

$$\|Tf_n\|_2 \to 1$$

By Banach-Alaoglu and the reflexivity and separability of $L^2(\mathbb{R}^d)$, there exists a subsequence $\{f_{n_k}\}$ that converges weakly to some $f \in L^2(\mathbb{R}^d)$. Moreover,

$$\|f\|_2 \leq \liminf_{k\to\infty} \|f_{n_k}\|_2 = 1$$

By the compactness of the operator T, we have strong convergence of the sequence $\{Tf_{n_k}\}$ in $L^2(\mathbb{R}^d)$. In particular, the norms converge and we have

$$1 = \lim_{k\to\infty} \|Tf_{n_k}\|_2 = \|Tf\|_2 \leq \|T\|_{2\to 2}\|f\|_2 = \|f\|_2$$

This shows that $\|f\|_2 = \|Tf\|_2 = 1$. Hence there exists an $f \neq 0$ such that $\mathrm{supp}(f) \subset E$ and $\mathrm{supp}(\widehat{f}) \subset F$ as essential supports. We now construct an infinite family of linearly independent functions from f. Define inductively

$$S_0 := \mathrm{supp}(f), \qquad S_1 := S_0 \cup (S_0 - x_0)$$

and

$$S_{k+1} := S_k \cup (S_k - x_k)$$

for $k \geq 1$, where $x_k \in \mathbb{R}^d$ are chosen such that $|S_k| < |S_{k+1}| < |S_k| + 2^{-k}$. It is clear that $f_k := f(\cdot + x_k)$ are linearly independent, $\mathrm{supp}(\widehat{f_k}) \subset F$,

$$\mathrm{supp}(f_k) \subset G := \bigcup_{n=1}^{\infty} S_n, \quad \text{and} \quad |G| < \infty$$

for each $k \in \mathbb{N}$. Setting $T'f := \mathbb{1}_G(\mathbb{1}_F\widehat{f})^\vee$, we see that $T'f_k = f_k$ for each $k \in \mathbb{N}$.

By Lemma 5 applied to the Hilbert-Schmidt operator T' with kernel $K'(x, y) = \mathbb{1}_G(x)\mathbb{1}_F^\vee(x - y) \in L^2(\mathbb{R}^{2d})$ we get our contradiction. Hence $\|T\|_{2\to 2} < 1$ and for all $f \in L^2(\mathbb{R}^d)$, we have

$$\|Tf\|_2 \leq \|T\|_{2\to 2}\|f\|_2.$$

By Lemma 4 with $\|T\|_{2\to 2} = \rho$, this proves Theorem 3. $\square$

With this theorem in mind, one may ask whether "large" sets can give similar bounds. The answer to this comes in the following theorem.

Theorem 6: **[Logvinenko-Sereda, '73]** Suppose that a measurable set $E \subset \mathbb{R}^d$ satisfies the following "thickness" condition: There exists a $\gamma \in (0,1)$ such that

$$|E \cap B| > \gamma |B|$$

for all balls B of some fixed radius R^{-1} in $\mathbb{R}^d$. If $f \in L^2(\mathbb{R}^d)$ satisfies $\text{supp}(\widehat{f}) \subset B(0,R)$, then

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(E)}$$

where $C = C(\gamma, d)$.

The following proof highlights the interesting interplay between real and complex analysis often found in modern harmonic analysis. In particular, theorems from multiple complex variables are heavily used in this proof.

Proof 6: By dilating $\mathbb{R}^d$, we may take $R = 1$. Fix $A \geq 2$ to be some constant which will be picked later. By an identical argument to the proof of Theorem 1, $f \in \mathcal{C}^\infty(\mathbb{R}^d)$. Partition $\mathbb{R}^d$ into cubes $\{Q\}$ of side length 2, and define the family of "good" cubes

$$\mathcal{G} := \{Q : \forall \alpha,\ \|\partial^\alpha f\|_{L^2(Q)} \leq A^{|\alpha|}\|f\|_{L^2(Q)}\}$$

Let $\mathcal{B} := \{Q : Q \notin \mathcal{G}\}$ be the set of "bad" cubes. We have the following L^2 estimate on these bad cubes:

$$\|f\|_{L^2(\bigcup_{\mathcal{B}} Q)} \leq CA^{-1}\|f\|_2, \tag{6}$$

where C is a universal constant. To prove this, consider the following sum:

$$\sum_{\alpha \neq 0}\sum_{\mathcal{B}} A^{-2|\alpha|}\|\partial^\alpha f\|^2_{L^2(Q)} \tag{7}$$

For each $Q \in \mathcal{B}$, there exists some multiindex α such that $\|\partial^\alpha f\|_{L^2(Q)} \geq A^{|\alpha|}\|f\|_{L^2(Q)}$. Hence (7) is bounded below by

$$\sum_{\mathcal{B}} \|f\|^2_{L^2(Q)} = \|f\|^2_{L^2(\bigcup_{\mathcal{B}} Q)}.$$

Moreover, by Theorem 1, we have an upper bound on (7) of the form

$$\begin{aligned}\sum_{\alpha\neq 0} A^{-2|\alpha|}\|\partial^\alpha f\|_2 &\leq \sum_{\alpha \neq 0} A^{-2|\alpha|}(2\pi)^{2|\alpha|}\|f\|_2^2 \\ &\leq \|f\|_2^2 \sum_{k=1}^{\infty}(k+1)^d A^{-2k}(2\pi)^{2k} \\ &\leq CA^{-2}\|f\|_2^2\end{aligned}$$

All together,

$$\|f\|^2_{L^2(\bigcup_{\mathcal{B}} Q)} \leq \sum_{\alpha\neq 0}\sum_{\mathcal{B}} A^{-2|\alpha|}\|\partial^\alpha f\|^2_{L^2(Q)} \leq CA^{-2}\|f\|_2^2$$

and taking square roots gets (6).

We claim the cubes $\mathcal{G}$ have the following nice property: For each $Q \in \mathcal{G}$ there exists an $x_0 \in Q$ such that for all α,

$$|\partial^\alpha f(x_0)| \leq A^{2|\alpha|+1} \|f\|_{L^2(Q)}. \tag{8}$$

To prove this, fix $Q \in \mathcal{B}$ and assume for each $x \in Q$, there exists an $\alpha(x)$ for which (8) does not hold. Then we have for each $x \in Q$,

$$\|f\|_{L^2(Q)} \leq A^{-2|\alpha|-1} |\partial^{\alpha(x)} f(x)| \leq \sum_\alpha A^{-2|\alpha|-1} |\partial^\alpha f(x)|.$$

Squaring this and integrating over Q,

$$\begin{aligned} |Q| \|f\|^2_{L^2(Q)} &\leq \int_Q \sum_\alpha A^{-4|\alpha|-2} |\partial^\alpha f(x)|^2 dx \\ &= \sum_\alpha A^{-4|\alpha|-2} \|\partial^\alpha f\|^2_{L^2(Q)} dx \\ &\leq \sum_\alpha A^{-2|\alpha|-2} \|f\|^2_{L^2(Q)} dx \\ &\leq A^{-2} \|f\|^2_{L^2(Q)} \sum_{k=0}^\infty (k+1)^d A^{-2k} \\ &\leq A^{-2} \|f\|^2_{L^2(Q)} \sum_{k=0}^\infty (k+1)^d 2^{-2k} \\ &= C A^{-2} \|f\|^2_{L^2(Q)}. \end{aligned}$$

This is a contradiction for large enough A, proving our claim.

Finally, we claim that there exists an $\eta > 0$, $\eta = \eta(d, \gamma, A)$ such that

$$Q \in \mathcal{G} \quad \Rightarrow \quad \|f\|_{L^2(E \cap Q)} \geq \eta \|f\|_{L^2(Q)}. \tag{9}$$

Again, we argue by contradiction. If our claim is false, then there exists $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ such that $\text{supp}(\widehat{f_n}) \subset B(0,1)$, measurable sets $\{E_n\}_{n\in\mathbb{N}}$ such that

$$|E_n \cap B(x,1)| \geq \gamma |B(x,1)| \quad \text{for all } x \in \mathbb{R}^d,$$

and "good" cubes $\{Q_n\}_{n\in\mathbb{N}}$ such that (by a possible rescaling)

$$\|f_n\|_{L^2(Q_n)} = 1, \quad \text{and} \quad \eta_n := \|f_n\|_{L^2(E_n \cap Q_n)} \to 0$$

as $n \to \infty$. By a possible translation (which leaves the support of $\widehat{f_n}$ unchanged), we may take $Q_n = [-1,1]^d =: Q_0$.

In order to furnish a contradiction, we wish to extract a convergent subsequence from our $\{f_n\}$. To do this, we first apply the Paley-Wiener theorem to each f_n to get a holomorphic extension $F_n : \mathbb{C}^d \to \mathbb{C}$. As in the single complex variable case, F_n is analytic and satisfies

$$F_n(z) = \sum_\alpha \frac{F_n^{(\alpha)}(x)}{\alpha!}(z-x)^\alpha$$

for any $x, z \in \mathbb{C}^d$, where we are using the standard multiindex conventions. By our earlier claim, for each Q_n there exists a point $x_n \in Q_n$ which satisfies

$$|\partial^\alpha f(x_n)| \leq A^{2|\alpha|+1} \|f\|_{L^2(Q_n)} = A^{2|\alpha|+1}$$

for all α. Hence, since $x_n \in \mathbb{R}^d \subset \mathbb{C}^d$, we have $F_n^{(\alpha)}(x_n) = \partial^\alpha f_n(x_n)$ and therefore

$$\begin{aligned} |F_n(z)| &\leq \sum_\alpha \frac{|F_n^{(\alpha)}(x)|}{\alpha!}|z-x|^\alpha \\ &\leq \sum_\alpha \frac{A^{2|\alpha|+1}}{\alpha!}|z-x|^\alpha \\ &\leq \sum_\alpha \frac{A^{2|\alpha|+1}}{\alpha!}(2+|z|)^\alpha \end{aligned}$$

For any $R > 0$, we thus have the following estimate for $|z| \leq R$;

$$|F_n(z)| \leq \sum_\alpha \frac{A^{2|\alpha|+1}}{\alpha!}(2+R)^\alpha \leq C(A, d, R).$$

This shows that our family $\{F_n\}_{n\in\mathbb{N}}$ is uniformly bounded. As in the single variable case, Montel's Theorem shows that this family is normal, allowing us to extract a locally uniform limit $F_{n_k} \to F_\infty$ which is also entire. Hence $\|F_\infty\|_{L^2(Q_0)} = 1$ by, for example, the bounded convergence theorem.

Define

$$\lambda_k := \eta_{n_k}\left(\frac{\gamma}{2}|Q_0|\right)^{-1/2}$$

and note by Chebyshev's inequality that

$$\begin{aligned} |\{x \in Q_0 \cap E_k : |f_{n_k}(x)| \geq \lambda_k\}| &\leq \lambda_k^{-2}\|f_{n_k}\|^2_{L^2(E_n\cap Q_0)} \\ &= \lambda_{n_k}^{-2}\eta_n^2 \\ &\leq \frac{\gamma}{2}|Q_0|. \end{aligned}$$

If we set

$$S_k := \{x \in Q_0 \cap E_{n_k} : |f_{n_k}(x)| \leq \lambda_k\}$$

then by the thickness condition,

$$\begin{aligned}|S_k| &= |E_{n_k} \cap Q_0| - |\{x \in Q_0 \cap E_{n_k} : |f_{n_k}(x)| \geq \lambda_k\}| \\ &\geq |E_{n_k} \cap B(0,1)| - |\{x \in Q_0 \cap E_{n_k} : |f_{n_k}(x)| \geq \lambda_k\}| \\ &\geq \gamma |B(0,1)| - \frac{\gamma}{2}|Q_0| \\ &\geq \frac{\gamma}{2}|B(0,1)|.\end{aligned}$$

Setting $S_\infty := \limsup_k S_k$ and by applying, say, Reverse Fatou's Lemma, we get

$$|S_\infty| \geq \frac{\gamma}{2}|B(0,1)| > 0.$$

However, since $\eta_n \to 0$, we also have that $\lambda_k \to 0$. Hence

$$S_\infty \subset \{z \in \mathbb{C}^d \cap Q_0 : F_\infty(z) = 0\}.$$

and so the zero set of F_∞ has positive measure. In particular, it is uncountable, and since Q_0 is compact, there is an accumulation point. As in the case for single variable holomorphic functions, this implies that $F_\infty \equiv 0$ on Q_0. This contradicts $\|F_\infty\|_{L^2(Q_0)} = 1$ and proves our claim.

Now with (9) in hand, we can finish the proof. We have

$$\begin{aligned}\|f\|^2_{L^2(E \cap \bigcup_{\mathcal{G}} Q)} &= \sum_{\mathcal{G}} \|f\|^2_{L^2(E \cap Q)} \\ &\geq \sum_{\mathcal{G}} \eta^2 \|f\|^2_{L^2(Q)} \\ &= \eta^2 \|f\|^2_{L^2(\bigcup_{\mathcal{G}} Q)} \\ &= \eta^2 (\|f\|_2^2 - \|f\|^2_{L^2(\bigcup_{\mathcal{B}} Q)})\end{aligned}$$

By (6), our bound on the bad cubes, we get that this last term is bounded below by

$$\eta^2(\|f\|_2^2 - C^2 A^{-2}\|f\|_2^2) \geq \frac{\eta^2}{2}\|f\|_2^2$$

for A large enough. This finishes the proof. □

By taking complements, we get a similar result for "small" or "thin" sets.

Corollary 7: Assume the measurable set $F \subset \mathbb{R}^d$ has the following "thinness" condition: Suppose there exists an $r > 0$ and $\beta \in (0,1)$ such that

$$|F \cap B(x,r)| < \beta |B(x,r)|$$

for all $x \in \mathbb{R}^d$. Then there exists an $\epsilon > 0$ such that either

$$\|\widehat{f}\|_{L^2(B(0,1)^c)} \geq \epsilon \|f\|_2 \quad \text{or} \quad \|f\|_{L^2(F^c)} \geq \epsilon \|f\|_2 \quad .$$

for all $f \in L^2(\mathbb{R}^d)$.

Proof 7: Let $E := F^c$. Then

$$|E \cap B(x,r)| = |B(x,r)| - |F \cap B(x,r)| < (1-\beta)|B(x,r)|,$$

and hence the set E is "thick" in the sense of Theorem 6. Let $f \in L^2(\mathbb{R}^d)$ and write

$$f = (\mathbb{1}_{B(0,1)}\widehat{f})^\vee + (\mathbb{1}_{B(0,1)^c}\widehat{f})^\vee =: f_1 + f_2$$

By Theorem 6, there exists a constant C depending only on β and d such that

$$\|f_1\|_2 \leq C\|f_1\|_{L^2(E)}$$

Fix $0 < \epsilon < C^{-1}/2$. Then if we have $\|f_2\|_2 \leq \epsilon\|f\|_2$

$$\begin{aligned}\|f\|_{L^2(E)} &\geq \|f_1\|_{L^2(E)} - \|f_2\|_{L^2(E)} \\ &\geq C^{-1}\|f\|_2 - \epsilon\|f\|_2 \\ &> \epsilon\|f\|_2\end{aligned}$$

If instead $\|f_2\|_2 \geq \epsilon\|f\|_2$, then

$$\|\widehat{f}\|_{L^2(B(0,1)^c)} = \|f_2\|_2 \geq \epsilon\|f\|_2$$

This finishes up the proof. □

With these results in hand, we are finally ready to tackle the Malgrange-Ehrenpreis Theorem. First we start with some definitions.

Definition 8: Let a polynomial $p \in \mathbb{C}[x_1, \ldots, x_d]$ be given. Write

$$p(\xi) = \sum_\alpha a_\alpha \xi^\alpha$$

with the standard multiindex conventions. Define $p(D) : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ by

$$p(D) := \sum_\alpha a_\alpha (2\pi i)^{-|\alpha|}\partial^\alpha$$

where the ∂^α denotes the distributional derivative. We have chosen this definition such that

$$\widehat{p(D)f}(\xi) = p(\xi)\widehat{f}(\xi)$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 9: [Malgrange-Ehrenpreis, '55] Let $\Omega \subset \mathbb{R}^d$ be bounded, connected, and open, and let $p \in \mathbb{C}[x_1, \ldots, x_d]$ be a nonzero polynomial. Then for all $g \in L^2(\Omega)$, there exists an $f \in L^2(\Omega)$ such that

$$p(D)f = g \quad \text{in distribution.}$$

That is, for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$ we have

$$\langle \varphi, g\rangle = \langle \overline{p}(D)\varphi, f\rangle.$$

As we shall see, the above theorem may be reduced to showing a particular operator is bounded and applying Hilbert space theory to obtain a solution. This motivates the following proposition.

Proposition 10: One has the bound

$$\|\overline{p}(D)\varphi\|_2 \geq C^{-1}\|\varphi\|_2$$

for all $\varphi \in \mathcal{C}_c^\infty(B(0,1))$, where $C = C(p,d)$.

To prove this proposition, we need two lemmas.

Lemma 11: Let p be a nonzero polynomial in $\mathbb{R}^d$. Then there exists $\beta = \beta(p,d) \in (0,1)$ such that

$$\left|\{x \in B : 2|p(x)| \leq \max_B |p|\}\right| < \beta|B|$$

for all balls B of radius 1.

Proof 11: Let N be the degree of the polynomial p. Fix the ball $B = B(0,1)$ with radius 1. Since the space of all polynomials on B of degree N or less is a finite dimensional vector space, all norms on it are comparable. Hence there exists a constant $C = C(N)$ such that for all q polynomials of degree N or less,

$$\max_{x\in B}(|q(x)| + |\nabla q(x)|) \leq C \max_{x\in B} |q(x)|.$$

In particular,

$$\max_{x\in B} |\nabla q(x)| \leq C \max_{x\in B} |q(x)|$$

By the mean value theorem applied to the convex set B, we have

$$|q(x) - q(y)| \leq \max_B |\nabla q||x-y| \leq C \max_B |q||x-y|.$$

This implies that for $|q(x_{\max})| = \max_B |q|$, with $x_{\max} \in \overline{B}$,

$$|q(x_{\max})| - |q(y)| \leq C \max_B |q||x_{\max} - y|$$

or equivalently

$$|q(y)| \geq |q(x_{\max})| - C \max_B |q||x_{\max} - y|.$$

If $y \in B(x_{\max}, 2C^{-1}) \cap B$, we have

$$|q(y)| > \frac{1}{2}\max_B |q|$$

and hence

$$B(x_{\max}, 2C^{-1}) \cap B \subset \{x \in B :\, 2|q(x)| \geq \max_B |q|\}.$$

Clearly, for any $x \in \overline{B}$ we have a lower bound of

$$0 < c|B| \leq |B(x, 2C^{-1}) \cap B|$$

for $c = c(d, N) \in (0, 1)$. Combining this with the above gets the lower bound

$$c|B| \leq \left|\{x \in B :\, 2|q(x)| \geq \max_B |q|\}\right|.$$

Setting $\beta = 1 - c$ gets

$$\left|\{x \in B :\, 2|q(x)| \leq \max_B |q|\}\right| \leq \beta|B|.$$

Since p is of degree N, the above inequality applies to p. Since for any $x_0 \in \mathbb{R}^d$, $p(x + x_0)$ is also a polynomial of degree N, the above estimate holds for all balls B of radius 1. This completes the proof. $\square$

Lemma 12: Let p be any nonzero polynomial n $\mathbb{R}^d$. Then there exists $\epsilon_0 = \epsilon_0(p) > 0$ such that

$$\left|\{x \in B :\, |p(x)| \leq \epsilon_0\}\right| \leq \beta|B|$$

where $\beta = \beta(d, p)$ is the same as in Lemma 11.

Proof 12: By Lemma 11, it suffices to show there exists $\epsilon_0 > 0$ such that

$$\inf_B \sup_{x \in B} |p(x)| \geq \epsilon_0$$

where the infimum is taken over all unit balls B. To see this, given any ball B of radius 1,

$$\left|\{x \in B :\, |p(x)| \leq \epsilon_0\}\right| \leq \left|\{x \in B :\, |p(x)| \leq \sup_{x \in B} |p(x)|\}\right| \leq \beta|B|.$$

To show there exists such an ϵ_0, fix $B = B(0, 1)$. As in the previous lemma, let N denote the degree of p and note that the space of all polynomials on $B(0, 1)$ of degree N is a finite dimensional space. All norms on this space are equivalent, so by comparing the polynomial norm to the L^∞ norm, we get

$$\max_B |q| \geq c \sum_\alpha |a_\alpha(q)|$$

where $c = c(N) > 0$ and $a_\alpha(q)$ are the coefficients of the polynomial q of degree less than or equal to N. Trivially, we have

$$\sum_\alpha |a_\alpha(q)| \geq \sum_{|\alpha| = N} |a_\alpha(q)|$$

so that

$$\max_B |q| \geq c \sum_{|\alpha| = N} |a_\alpha(q)|.$$

Setting $q(x) = p(x + x_0)$, this implies

$$\max_{B-x_0} |p| \geq c \sum_{|\alpha|=N} |a_\alpha(q)| = c \sum_{|\alpha|=N} |a_\alpha(p)|,$$

where the last inequality follows from the translation invariance of the coefficients of the highest powers of p. Setting

$$\epsilon_0 := c \sum_{|\alpha|=N} |a_\alpha(p)| > 0$$

we get our inequality. □

With Lemma 12 in hand, it is simple to prove the proposition. The main idea of the proof is to apply the "thin" condition in Corollary 7 to obtain nice lower bounds.

<u>Proof 10:</u> Set

$$F := \{x \in \mathbb{R}^d : |p(x)| \leq \epsilon_0\}$$

for ϵ_0 chosen as in Lemma 12. Corollary 7 shows that F satisfies the "thin" condition, so there exists an $\epsilon_1 = \epsilon_1(d) > 0$ such that either

$$\|\varphi\|_{L^2(B(0,1)^c)} \geq \epsilon_1 \|\varphi\|_2 \quad \text{or} \quad \|\widehat{\varphi}\|_{L^2(F^c)} \geq \epsilon_1 \|\varphi\|_2$$

for all $\varphi \in L^2(\mathbb{R}^d)$. In particular, for nonzero $\varphi \in \mathcal{C}_c^\infty(B(0,1)) \hookrightarrow L^2(\mathbb{R}^d)$, we have $\|\varphi\|_{L^2(B(0,1)^c)} = 0$, so that the second inequality must hold:

$$\|\widehat{\varphi}\|_{L^2(F^c)} \geq \epsilon_1 \|\varphi\|_2.$$

Putting this inequality with Plancherel's theorem, we have that

$$\begin{aligned} \|\overline{p}(D)\varphi\|_2 &= \|\overline{p}\widehat{\varphi}\|_2 \\ &\geq \|\overline{p}\widehat{\varphi}\|_{L^2(F^c)} \\ &\geq \epsilon_0 \|\widehat{\varphi}\|_{L^2(F^c)} \\ &\geq \epsilon_0 \epsilon_1 \|\varphi\|_2 \end{aligned}$$

This completes the proof of the proposition. □

Finally, we can proceed to complete the proof of Theorem 9.

<u>Proof 9:</u> It is a simple functional analysis problem to complete the proof. First note that since Ω is bounded, we may take it to be contained in a ball. Since a distributional solution on this ball gives a distributional solution on Ω, we may assume that Ω is a ball. By translation and dilation symmetries, we may take this ball to be $B(0,1)$. Define

$$X := \left\{ \overline{p}(D)\varphi : \varphi \in \mathcal{C}_c^\infty(B(0,1)) \right\} \hookrightarrow L^2(\mathbb{R}^d).$$

Note first that X is a linear subspace of $L^2(\mathbb{R}^d)$. Define the linear operator

$$\ell : X \to \mathbb{C}$$

by $\ell(\overline{p}(D)\varphi) = \langle \varphi, g\rangle_2$. This is well defined, for if $\overline{p}(D)\varphi = \overline{p}(D)\psi$, we have by Proposition 10 that

$$0 = \|\overline{p}(D)(\varphi - \psi)\|_2 \geq C^{-1}\|\varphi - \psi\|_2$$

so that $\varphi = \psi$ in $L^2(\mathbb{R}^d)$. Moreover, ℓ is continuous, since by Cauchy-Schwarz and Proposition 10

$$|\ell(\overline{p}(D)\varphi)| \leq \|\varphi\|_2\|g\|_2 \leq (C\|g\|_2)\|\overline{p}(D)\varphi\|_2$$

So by Hahn-Banach, there exists an extension $\tilde{\ell} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, and by the Riesz Representation Theorem there exists an $f \in L^2(\mathbb{R}^d)$ such that

$$\tilde{\ell}(\cdot) = \langle \cdot, f\rangle_2.$$

Putting all this together, for any $\varphi \in \mathcal{C}_c^\infty(B(0,1))$, we have $\overline{p}(D)\varphi \in X$, so

$$\langle \overline{p}(D)\varphi, f\rangle_2 = \tilde{\ell}(\overline{p}(D)\varphi) = \ell(\overline{p}(D)\varphi) = \langle \varphi, g\rangle_2.$$

Therefore $p(D)f = g$ in distribution. □

References

[1] Muscalu C. and Schlag W. *Classical and Multilinear Harmonic Analysis*, volume 1. Cambridge University Press, 2016.