# DRP - DIFFERENTIAL TOPOLOGY

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These notes are meant to accompany the book *Topology from The Differential Perspective* by J. Milnor.

## 1. Smooth Manifolds and Smooth Maps

This section is mostly devoted to definitions and notations. You may assume all the facts from multivariable calculus on $\mathbb{R}^d$ (see appendix) for the exercises.

**Notations:**

1. The symbols $i, j, k, l, m, n$ will denote natural numbers.
2. The symbols $M, N, P$ are always smooth manifolds of dimension $m, n$, and $p$ respectively.
3. If $A \subset B$, then the notation $A \hookrightarrow B$ denotes the inclusion map.
4. The symbols $X, Y, Z$ always denote sets, and $U, V, W$ open sets.
5. If $x \in \mathbb{R}^n$, the notation $x_i$ denotes the $i$th component of $x$. Likewise, if $f : \mathbb{R}^k \to \mathbb{R}^l$, then $f_j(x)$ denotes the $j$th component of the vector $f(x)$.
6. We write $B_r(x) := \{y \in \mathbb{R}^n \mid \|y\| < r\}$, to denote the ball of radius $r$ centered at $x$.
7. We write $df_x$ to mean the derivative of $f$ at $x$.

**Definition 1.1:** **(Smooth Maps)** Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^l$, and $f : X \to Y$. We say that $f$ is *smooth* if for all $x \in X$, there exists some open set $U \subset \mathbb{R}^k$ containing $x$ and an extension $F : U \to \mathbb{R}^l$ of $f$ which is smooth in the regular sense.

Check for yourself that if $X, Y$ are open sets, then Definition 1 agrees with the regular definition of smooth.

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**<u>Definition 1.2:</u> (Diffeomorphism)** Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$. We say $f : X \to Y$ is a *diffeomorphism* if $f$ is a bijection, and the mappings $f : X \to Y$ and $f^{-1} : Y \to X$ are smooth maps in the sense of Definition 1. We say $X, Y$ are diffeomorphic if there exists a diffeomorphism between them.

**<u>Exercise 1.1:</u>** Show that the relation of two sets being diffeomorphic is an equivalence relation.

**<u>Definition 1.3:</u>(Smooth Manifold)** A set $M \subset \mathbb{R}^k$ is a *smooth manifold* of dimension $m$ if for every $x \in M$ there exists an open set $U \subset \mathbb{R}^k$ containing $x$, and open set $V \in \mathbb{R}^m$ such that $U \cap M$ is diffeomorphic to $V$.

This definition says that $M$ is *locally diffeomorphic to an open set of* $\mathbb{R}^m$. The word *local* means it is a property which holds for some neighborhood of each point. Definition 3 generalizes the idea of what a "surface" is in $\mathbb{R}^k$. Just as with the surfaces come across in multivariable calculus, we can "parametrize" manifolds, at least locally.

**<u>Definition 1.4:</u>** Let $M \subset \mathbb{R}^k$ be a smooth manifold of dimension $m$. We call the pair $(g, U)$ a parametrization around $x \in M$ if $U$ is open, $g : U \to g(U)$ is a diffeomorphism, $g(U) \subset M$, and $x \in g(U)$.

So, yet another way to phrase the definition of smooth manifold $M$ is to say that every point in $M$ can be parametrized. The following exercise gives some examples of smooth manifolds:

**<u>Exercise 1.2:</u>** Show that the following sets are smooth manifolds and identify their dimension.

1. $M = \{x \in \mathbb{R}^k | \quad x_{m+1} = \cdots = x_k = 0\}$. (That is, $\mathbb{R}^m$ when viewed as a subset of $\mathbb{R}^k$ is a smooth manifold.)

2. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a smooth map. Then consider the graph of $f$:
$$M = \{(x, y) \in \mathbb{R}^k \times \mathbb{R} | \quad f(x) = y\}.$$

3. $M = \{x \in \mathbb{R}^k | \quad x_1^2 + \cdots + x_k^2 = 1\}$. (hint: We have almost done the case where $k = 2$ already. Show the $k = 2$ case first, and then generalize.)

Sometimes, it is annoying to keep track of both $U$ and $V$ in the definition of smooth manifolds when doing proofs, so we like to use the following definition:

**<u>Definition 1.3':</u>** A set $M \subset \mathbb{R}^k$ is a *smooth manifold* of dimension $m$ if for every $x \in M$ there exists an open set $U \subset \mathbb{R}^k$ containing $x$ such that $U \cap M$ is diffeomorphic to $\mathbb{R}^m$.

**Exercise 1.3:** Show that Definition 3 and Definition 3' are equivalent by following these steps:

1. Show Definition 3' implies Definition 3
2. Let $M$ be a smooth manifold according to Definition 3. Let $x \in M$ and $g : U \cap M \to V$ be the diffeomorphism between the sets given Definition 3. Show by "shrinking" $U$ we can take $V$ to be an open ball. This technique of "shrinking" will be used heavily later on.
3. Show that the maps $M_\lambda : \mathbb{R}^m \to \mathbb{R}^m$ given by $M_\lambda(y) = \lambda y$ and $\tau_{y_0} : \mathbb{R}^m \to \mathbb{R}^m$ given by $\tau_{y_0}(y) = y - y_0$ are diffeomorphisms for any $\lambda \in \mathbb{R} \setminus \{0\}$ and $y_0 \in \mathbb{R}^m$.
4. Show that this implies that we can take the ball in part 1 to be the unit ball $B_1(0) \subset \mathbb{R}^m$.
5. Construct a diffeomorphism $f : (-\pi/2, \pi/2) \to \mathbb{R}$ (hint: its a common function you know).
6. Using $f$, construct a diffeomorphism $F : B_1(0) \to \mathbb{R}^m$.

The last step in this exercise is a surprising result! It says that the unit ball can be smoothly deformed into the whole space.

While this result seems to imply that we all connected open subsets of $\mathbb{R}^m$ are diffeomorphic, this is in fact false. We shall see why later.

## 2. Tangent Spaces and Derivatives

Now, we wish to define a suitable definition for the derivative of a smooth mapping between smooth manifolds. As in the case where $f : U \to \mathbb{R}^l$ is a smooth map defined on an open set $U \subset \mathbb{R}^k$, we expect the derivative $df_x$ to be a linear operator. The following section is devoted to defining where the derivative is defined and where its image lives.

First, we consider the case where regular multivariable calculus applies. Let

$U \subset \mathbb{R}^k$ be an open set, and assume $f : U \to \mathbb{R}^l$ is smooth. Then clearly for any $x \in U$, the linear mapping $df_x$ takes $\mathbb{R}^k \to \mathbb{R}^l$. We thus make the following definition:

**Definition 2.1:** **(Tangent Space of an Open Set)** Let $U \subset \mathbb{R}^k$ be an open set. Then we define for $x \in U$ the *tangent space* of $U$ at $x$ to be $\mathbb{R}^k$, written $TU_x = \mathbb{R}^k$.

If we have a smooth map $f : U \to \mathbb{R}$, consider its graph. That is, the set

$$\text{graph}(f) := \{(x, f(x)) \,|\, x \in U\}.$$

This can be visualized as a "sheet" above $U$, where every point in $U$ is mapped by $f$ to a point above or below $U$. The tangent space $TU_x$, even though it is just $\mathbb{R}^k$, is often thought of as the plane tangent to the graph of $f$ at $x$, hence the name. Let's try and get familiar with this picture.

**Exercise 2.1:** Draw the tangent plane as described above for the following functions:

1. Let $f : (-1, 1) \to \mathbb{R}$ be given by $f(x) = \sqrt{1 - x^2}$. Draw the tangent plane at the points $x = -1/2, 0, 1/2$.

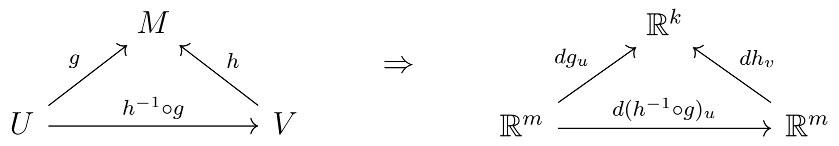
2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x) = x_1^2 + x_2^2$. Draw the tangent plant at the point $x = 0$.

Now, we wish to extend our definition of tangent planes to smooth manifolds. A natural idea would be to take a parametrization of our manifold $M$ (i.e. a smooth diffeomorphism from an open set in $\mathbb{R}^m$ to a neighborhood of $M$) and draw a tangent plane to its graph. Convince yourself this idea makes sense by considering, for example, the circle $\mathbb{S}^1 \subset \mathbb{R}^2$.

Indeed, this is how we define tangent spaces of smooth manifolds, but in its current form it has some problems. The first problem is that we can parametrize our manifold in many different ways. How are we sure that the "tangent space" as we want to define it doesn't depend on our parametrization? The second problem is what dimension is our "tangent space"? Does it align with our picture? These issues are fixed using the following propositions:

**Proposition 2.1:** Let $M \subset \mathbb{R}^k$ be a smooth manifold of dimension $m$. Let $x \in M$, and consider any two parametrizations $(g, U), (h, V)$ around $x$. Let $u = g^{-1}(x)$ and $v = h^{-1}(x)$ for some $u \in U$ and $v \in V$. Then $\text{Image}(dg_u) = \text{Image}(dh_v)$.

**Proof Sketch:** First note that the derivatives $dg_u, dh_v$ are well defined. Next, by possibly shrinking $U$, we may assume that $g(U) \subset h(V)$. Thus the composition $h^{-1} \circ g$ is well defined. By the chain rule we have the following commutative diagrams




(A commutative diagram is just a shorthand way of writing that a series of maps composed together is equal.) Since the composition $h^{-1} \circ g$ is a diffeomorphism, $d(h^{-1} \circ g)$ is and one-to-one and onto. This shows that the image of $dg_u$ is the same as $dh_v$.

**Exercise 2.2:** Turn the above proof sketch into an actual proof by rigorously explaining the following:

1. Explain why $dg_x$ and $dh_x$ are well defined even though $g$ and $h$ take points in $\mathbb{R}^m$ to points in a manifold (hint: where does our manifold live?)

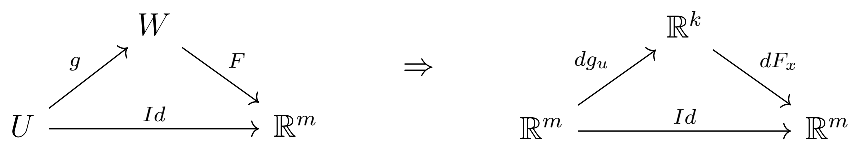
2. Prove rigorously why we can "shrink" $U$ to obtain $g(U) \subset h(V)$. Explain why "shrinking" $U_1$ does not change the image of $dg_u$.

3. Prove that if $f$ is a diffeomorphism between two open sets $W_1 \subset \mathbb{R}^k$ and $W_2 \subset \mathbb{R}^l$, then for any $w \in W_1$, we have $df_w : \mathbb{R}^k \to \mathbb{R}^l$ is one-to-one and onto.

4. Complete the proof by showing this implies that Image($dg_u$) = Image($dh_v$).

**Proposition 2.2:** Let $M \subset \mathbb{R}^k$ be a smooth manifold of dimension $m$. Then for any $x \in M$ and any parametrization $(g, U)$ around $x = g^{-1}(u)$, we have that Image($dg_x$) is an $m$-dimensional vector space.

**Proof:** By definition of diffeomorphism, we know $g^{-1} : g(U) \to U$ is smooth. By the definition of smooth, there exists a neighborhood $W$ of $x$ and a smooth function $F : W \to \mathbb{R}^m$ which agrees with $g^{-1}$ on $g(U) \cap W$. Again, by shrinking $U$, we may take $g(U) \subset W$, so the composition $F \circ g$ is well defined. In fact, since $F = g^{-1}$ on $g(U)$, we have that $F \circ g = Id$ on $g(U)$. We therefore have the following commutative diagrams:



This implies that $dg_u$ has full rank when viewed as a matrix. That is, $\dim(\text{Image}(dg_u)) = m$.

By our two propositions, we can therefore give the following definition:

**Definition 2.2:** **(Tangent Space of a Manifold)** Let $M \subset \mathbb{R}^k$ be a manifold of dimension $m$. We define the tangent space of $M$ at $x$ to be the vector space

$$TM_x := \text{Image}(dg_u)$$

where $(g, U)$ is any parametrization around $x$ with $u = g^{-1}(u)$. We have showed that this set does not depend on our parametrization and is of vector space dimension $m$.

Now that we have defined our tangent spaces for manifolds, we have enough language to give the following definition of derivative. The idea for the definition is simple: define derivative the only way we know how.

**Definition 2.3:** **(Derivative between Manifolds)** Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$ be smooth manifolds of dimensions $m$ and $n$. If $f : M \to N$ is a smooth mapping, we define for $x \in M$ and $y = f(x)$ the derivative of $f$ at $x$ to be the linear map

$$df_x : TM_x \to TN_y$$

constructed in the following way: Since $f$ is smooth, there exists a neighborhood $W \subset \mathbb{R}^k$ and a smooth $F : W \to \mathbb{R}^l$ that agrees with $f$ on $W \cap M$. Then we set $df_x := dF_x$.

As with the definition of tangent spaces of manifolds, we come across problems with this definition. The following proposition takes care of these problems.

**Proposition 2.3:** Let us use the same notation as in Definition 2.3. Then $df_x$ does not depend on our choice of $F$, $\text{Image}(dF_x) \subset TN_y$, and $\text{Domain}(dF_x) \subset TM_x$.

**Proof:** Let $(g, U)$ be a parametrization around $x$ in $M$ and $(h, V)$ be a parametrization around $y$ in $N$. By shrinking $U$, we may assume that $g(U) \subset W$ and that $f \circ g(U) \subset h(V)$. Then we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{F} & \mathbb{R}^l \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h} \\ U & \xrightarrow{h^{-1}\circ f\circ g} & V \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathbb{R}^k & \xrightarrow{dF_x} & \mathbb{R}^l \\ {\scriptstyle dg_u}\uparrow & & \uparrow{\scriptstyle dh_v} \\ \mathbb{R}^m & \xrightarrow{d(h^{-1}\circ f\circ g)_u} & \mathbb{R}^n \end{array}$$

where $u = g^{-1}(x)$ and $v = h^{-1}(y)$. That is,

$$\boxed{dF_x = dh_v \circ d(h^{-1} \circ f \circ g)_u \circ (dg_u)^{-1}}$$

This immediately shows that Image$(dF_x) \subset TN_y$ and Domain$(dF_x) \subset TM_x$ (why?). Moreover, the right hand side of the above equation does not depend on our particular choice of $F$ for the definition of smoothness of $f$, completing our proof.

It is interesting to note that the boxed equation is actually how most books define $df_x$, i.e. in terms of parametrizations.

Unlike in the multivariable or single variable calculus case, we often do not care about what the derivative $df_x$ actually is. The boxed equation above gives a way to compute it, but there isn't much use for it other than to show the following properties:

<u>**Exercise 2.3:**</u>

1. Let $f : M \to N$ and $g : N \to P$ be smooth mappings between smooth manifolds. Show that if $x \in M$ and $y = f(x)$, we have
$$d(g \circ f)_x = dg_y \circ df_x$$
or equivalently the following diagram commutes:
$$TM_x \xrightarrow{df_x} TN_y \xrightarrow{dg_y} TP_z, \qquad TM_x \xrightarrow{d(g\circ f)_x} TP_z$$
where $z = g(y)$.

2. Show that if $f : M \to N$ is a diffeomorphism, then for any $x \in M$ with $y = f(x)$, $df_x : TM_x \to TN_y$ is one-to-one and onto.

## 3. Regular Points and Regular Values

Next, we will discuss the topic of regular values. For a smooth function $f : M \to N$ between manifolds, a regular value of $f$ is a point in $N$ which behaves nicely under the pullback of $f$. We define it as the following:

<u>**Definition 3.1:**</u> Let $f : M \to N$ be a smooth mapping between manifolds of the same dimension. Then a *regular point* is any $x \in M$ such that $df_x$ is one-to-one and onto. A *regular value* of $f$ is any point $y \in N$ such that $f^{-1}(\{y\})$ contains only regular values.

We also want a name for points and values which are not regular, so we have the following definition also:

**Definition 3.2:** Let $f : M \to N$ be a smooth mapping between two manifolds of the same dimension. Then a *critical point* of $f$ is any $x \in M$ that is not a regular value (or equivalently, $df_x$ is not one-to-one or not onto). Likewise, a *critical value* of $f$ is any $y \in N$ which is not a regular value.

Note that as in the $f : \mathbb{R}^n \to \mathbb{R}^n$ case, the inverse function theorem (see *Appendix*) holds for mappings between manifolds of the same dimension:

**Theorem 3.1:** **(Inverse Function Theorem)** Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$, and let $f : M \to N$ is a smooth mapping between manifolds of the same dimension. Then if $x \in M$ is a regular point of $f$, there exists a neighborhood $U \subset \mathbb{R}^k$ of $x$ such that $f : U \cap M \to f(U \cap M)$ is a diffeomorphism.

**Proof:** Let $x \in M$ be a regular point of $f$. Then for $y = f(x)$, $df_x : TM_x \to TN_y$ is bijective. Let $(g, U)$ be a parametrization around $x$ in $M$ and $(h, V)$ be a parametrization around $y$ in $N$. Call $u = g^{-1}(x)$ and $v = h^{-1}(y)$. We may assume that $f \circ g(U) \subset V$.

Consider $h^{-1} \circ f \circ g : U \to \mathbb{R}^m$. We have by applying Exercise 2.3 twice that

$$d(h^{-1} \circ f \circ g)_u = dh_y^{-1} \circ df_x \circ dg_u$$

Moreover, Exercise 2.3 shows that $dh_y^{-1}$ and $dg_u$ are both bijective, and since we assumed $df_x$ is also, the composition $dh_y^{-1} \circ df_x \circ dg_u$ is bijective. This shows by the regular inverse function theorem that the mapping $h^{-1} \circ f \circ g$ maps a neighborhood $U'$ around $u$ diffeomorphically to a neighborhood $V'$ around $v$. This shows that the composition

$$h \circ (h^{-1} \circ f \circ g) \circ g^{-1} = f$$

maps $g(U')$ diffeomorphically to $h(V')$. We have proven our theorem.

Now, we have the following nice condition for regular values (assuming our domain is compact).

**Lemma 3.1:** Let $f : M \to N$ be a smooth mapping between two manifolds of the same dimension. If $M$ is compact, then for any regular value $y \in N$, $f^{-1}(\{y\})$ contains finitely many points.

**Proof:** Since $\{y\} \in N$ is closed and $f$ is continuous, $f^{-1}(\{y\}) \subset M$ is closed. Since a closed subset of a compact set is also compact, we know that $f^{-1}(\{y\})$ is compact. Moreover by Theorem 3.1, for any $x \in f^{-1}(\{y\})$, we know that there exists a neighborhood $U_x$ of $x$ such that $f : U_x \cap M \to f(U_x \cap M)$ is a diffeomorphism. In particular, it is one-to-one on $U_x \cap M$, so only the point $x$ gets mapped to $y$ in this neighborhood (drawing pictures is helpful).

Now, we use the property of compactness to say there are only finitely many of these neighborhoods. First note that

$$f^{-1}(\{y\}) \subset \bigcup_{x \in f^{-1}(\{y\})} U_x$$

since each $U_x$ contains $x$. Now since we showed before that $f^{-1}(\{y\})$ was compact, by the definition of compactness we can pick $x_1, \ldots, x_n$ such that

$$f^{-1}(\{y\}) \subset \bigcup_{k=1}^{n} U_{x_k}.$$

In particular, each $U_{x_k}$ contains only one point of $f^{-1}(\{y\})$, so we know that there are at most $n$ points in $f^{-1}(\{y\})$. This completes the proof.

From this lemma, we know that it makes sense to count the number of points in $f^{-1}(\{y\})$. Let's give this number a name.

**Definition 3.3:** Let $f : M \to N$ be a smooth map between manifolds of the same dimension. Assume $M$ is compact, and define

$$RV(f) := \{y \in N|\ y \text{ is a regular value of } f\}$$

Then define the mapping $\#f^{-1} : RV(f) \to \mathbb{N}$ by

$$\#f^{-1}(y) = \left|f^{-1}(\{y\})\right|.$$

This function above turns out to be very important in understanding the structure of smooth compact manifolds (and sometimes even non-compact ones). In particular, it will give us our first *topological invariant*. We list an important property of the function below.

**Proposition 3.1:** If $f : M \to N$ is a smooth mapping between manifolds of the same dimension, then the map $\#f^{-1}$ defined above is *locally constant*. That is, for every point $y \in RV(f)$, there is a neighborhood around $y$ such that $\#f^{-1}$ is constant on that neighborhood.

**Proof:** Let $y \in RV(f)$. Then by Lemma 3.1 there are finitely many distinct points $x_1, \ldots, x_n \in f^{-1}(\{y\})$. Now, we can cover the points $x_1, \ldots, x_n$ by disjoint open neighborhoods $U_1, \dots, U_n$ such that $f$ sends each $U_k$ diffeomorphically to $f(U_k) = V_k$. Note that the set

$$V = \bigcap_{k=1}^{n} V_k \setminus f\Big(M \setminus (\bigcup_{k=1}^{n} U_k)\Big)$$

satisfies that for any $v \in V \cap RV(f)$, $\#f^{-1}(v) = \#f^{-1}(y)$. This proves the proposition.

## Appendix: Multivariable Calculus

Here, we review some facts from multivariable calculus. We start as usual with some definitions:

**Definition:** **(Smooth Functions)** Let $U \subset \mathbb{R}^k$ be an open set. We say that a function $f : U \to \mathbb{R}^l$ is *smooth* if every partial derivative exists and is continuous.

We will not prove it, but if $f : U \to \mathbb{R}^l$ is smooth, then for every $x \in U$ and $h \in \mathbb{R}^k$, the limit

$$\lim_{t \to 0} \frac{1}{t}[f(x + th) - f(x)]$$

exists and is finite. Note that this limit is identical to the definition of the derivative in one dimension, the only change being the arguments and limit are vectors. This leads to the following definition:

**Definition:** **(Derivative)** Let $U \subset \mathbb{R}^k$ be an open set and $f : U \to \mathbb{R}^l$ be a smooth map. For $x \in U$ we define the *derivative of* $f$ *at* $x$ to be the map $df_x$ given by

$$df_x(h) := \lim_{t \to 0} \frac{1}{t}[f(x + th) - f(x)].$$

Be particularly aware of what type of object $df_x$ is! It is not a number or a vector, but *a mapping of* $\mathbb{R}^k$ *to* $\mathbb{R}^l$. We list some important properties of this mapping.

**Proposition:(Properties of the Derivative)** Let $U \subset \mathbb{R}^k$ be open, $V \subset \mathbb{R}^l$ be open, $f : U \to \mathbb{R}^l$ be smooth, and $g : V \to \mathbb{R}^j$ be smooth. The following hold:

1. For any $x \in U$, the mapping $df_x : \mathbb{R}^k \to \mathbb{R}^l$ is linear.

2. For any $x \in U$ we can view $df_x$ as a matrix:

$$df_x(h) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \frac{\partial f_1}{\partial x_3}(x) & \cdots & \frac{\partial f_1}{\partial x_k}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & & & \frac{\partial f_2}{\partial x_k}(x) \\ \vdots & & & & \vdots \\ \frac{\partial f_l}{\partial x_1}(x) & & \cdots & & \frac{\partial f_l}{\partial x_k}(x) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_k \end{bmatrix}$$

3. If $f(U) \subset V$, then the composition $g \circ f$ is well defined, smooth, and for $x \in U$, we have
$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$
When viewed as matrices, the composition of derivatives above is just matrix multiplication.

This chain rule applied many times gives the following heuristic result: "Any commutative diagram of smooth maps gives rise to a commutative diagram of their derivatives."

<u>**Theorem:**</u> **(Inverse Function)** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function, and assume that $x \in \mathbb{R}^n$ such that $df_x$ is invertible (or equivalently, bijective). Then there exists a neighborhood $U \ni x$ such that $f : U \to f(U)$ is a diffeomorphism.